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On the growth of α-potentials in R^n and thinness of sets

Yoshihiro MIZUTA

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1. Introduction

In the n-dimensional euclidean space R^n, we define the α-potential of a nonnegative (Radon) measure μ by

$$R_\alpha\mu(x) = \int R_\alpha(x-y)d\mu(y),$$

where $R_\alpha(x)=|x|^{\alpha-n}$ if $0<\alpha<n$ and $R_n(x)=\log(1/|x|)$. Then it is easy to see that $|R_\alpha\mu| \not\equiv \infty$ if and only if

$$(1) \qquad \begin{aligned} &\int (1+|y|)^{\alpha-n}d\mu(y) < \infty \qquad \text{in case} \quad \alpha < n, \\ &\int \log(2+|y|)d\mu(y) < \infty \qquad \text{in case} \quad \alpha = n. \end{aligned}$$

Let h be a positive and nonincreasing function on the interval $(0, \infty)$ such that $h(r) \leqq$ const. $h(2r)$ for $r>0$. In this paper, we first discuss the behavior of $h(|x|)^{-1}R_\alpha\mu(x)$ at the origin, in connection with the growth of the mean value of $R_\alpha\mu$ over the open balls centered at the origin. In our discussions, the aim is to find a criterion of the exceptional set E for which $h(|x|)^{-1}R_\alpha\mu(x)$ has limit zero or remains bounded above as x tends to 0 outside E. Our results obtained below will be similar to the characterizations of minimal thinness ([4]), minimal semithinness ([5], [6]) and logarithmical thinness and semithinness ([7]).

The thinness can be defined in terms of the α-capacity, like the expression of Wiener's criterion (see e.g. Brelot [1] and Landkof [3]). In this paper, letting $B(x, r)$ denote the open ball with center at x and radius r, we define the α-capacity of a set E in $B(0, 2^{-1})$ by

$$C_\alpha(E) = \inf \mu(R^n),$$

where the infimum is taken over all nonnegative measures μ with support in $B(0, 1)$ such that $R_\alpha\mu(x) \geqq 1$ for every $x \in E$.

The exceptional set E appeared in the discussion will satisfy the condition that $h_i^{-1}\sum_{j=1}^{\infty} h_j \min\{a_i, a_j\}C_\alpha(E_j)$ is bounded or has limit zero as $i\to\infty$, where $h_j=h(2^{-j})$, $a_j=2^{j(n-\alpha)}$ if $\alpha<n$, $a_j=j$ if $\alpha=n$ and $E_j=E \cap B(0, 2^{-j})-B(0, 2^{-j-1})$. For particular choices of h, the condition means the α-thinness of E, the α-semi-