HIROSHIMA MATH. J.
16 (1986), 51–75

Lie algebras satisfying the weak minimal condition on ideals

Takanori SAKAMOTO

(Received January 21, 1985)

Introduction

A group is said to satisfy the weak minimal condition on subgroups if it has no infinite descending chains of subgroups in which all neighbouring indices are infinite. Groups satisfying such a condition were first studied by D. I. Zaĭcev [17].

The purpose of this paper is to introduce analogously in Lie algebras the weak minimal conditions (wmin) relaxing the minimal conditions, and to investigate the properties of Lie algebras satisfying the weak minimal conditions on various subalgebras. The aspects of Lie algebras satisfying the weak minimal conditions are not similar to those of groups satisfying the corresponding conditions. One of the main reasons seems to be the following: In group theory every subgroup of finite index contains a normal subgroup of finite index, while in the theory of Lie algebras a subalgebra of finite codimension does not necessarily contain an ideal of finite codimension. Moreover, we define the weak maximal conditions (wmax) and develop the results on them in the course of the study of the weak minimal conditions.

The main results of this paper are as follows.

(1) An ideally soluble, hypoabelian Lie algebra satisfying the weak minimal condition on ideals is soluble (Theorem 2.4).

(2) A non-abelian ideally finite Lie algebra satisfying the weak minimal condition on non-abelian 2-step subideals satisfies the minimal condition on ideals (Theorem 3.6).

(3) If $\mathfrak{X}_i$ $(i=1, 2, 3)$ is one of the classes of abelian, nilpotent and soluble Lie algebras, then the following conditions are equivalent: (a) wmin on $\mathfrak{X}_1$-subideals (resp. ascendant $\mathfrak{X}_1$-subalgebras); (b) wmax on $\mathfrak{X}_2$-subideals (resp. ascendant $\mathfrak{X}_2$-subalgebras); (c) the minimal condition on $\mathfrak{X}_3$-subideals (resp. ascendant $\mathfrak{X}_3$-subalgebras) (Theorem 4.2).

(4) Each of the following Lie algebras is finite-dimensional: (a) a nilpotent algebra satisfying wmin or wmax on abelian ideals; (b) a supersoluble algebra satisfying wmin or wmax on abelian 2-step subideals; (c) a hyperabelian algebra satisfying wmin or wmax on abelian 3-step subideals; (d) an $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$-algebra satisfying wmin or wmax on abelian subideals; (e) an $\acute{\mathrm{E}}\mathfrak{A}$-algebra satisfying wmin or wmax on abelian ascendant subalgebras; (f) an $\acute{\mathrm{E}}\mathrm{L}(\mathrm{wasc})(\mathrm{E}\mathfrak{A} \cup \mathfrak{F})$-algebra