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Spaces of orderings and quadratic extensions of fields

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Let P be a preordering of a field F of finite index and $K=F(\sqrt{a})$ be a radical extension of F(i.e. a is an element of Kaplansky's radical of F). We denote by n the number of the connected components of $X(F/P)$. In [4], we showed that $n=\dim H_F(P)/P$ ([4], Theorem 2.5) and the number of connected components of $X(K/P')$ is $2n$, where $P'=\Sigma P\dot{K}^2$ ([4], Theorem 3.10).

The main purpose of this paper is to study a relation between $X(F)$ and $X(K)$, where F is a quasi-pythagorean field whose Kaplansky's radical $R(F)$ is of finite index and $K=F(\sqrt{a})$ is a quadratic extension of F. In §2, we show that if $a\in H_F$, then $X(K)$ is equivalent to $H_F(a)\oplus H_F(a)$ (Theorem 2.9). In §3, we assume that $X(F)$ is connected and show that the following results. If $a\in B_{R(F)}$, then $X(K)$ is equivalent to $X(F)$, where $B_{R(F)}$ is the set of $R(F)$-basic elements of $\dot{F}$ (Theorem 3.3). If $a\in B_{R(F)}\setminus \pm R(F)$ and $D_F\langle 1, a\rangle D_F\langle 1, -a\rangle = B_{R(F)}$, then $X(F)$ is equivalent to a group extension of $H_{X_1}(a)\oplus H_{X_1}(a)$, where the space $H_{X_1}(a)$ is defined in §3 (Theorem 3.5).

§1. Valuations on quasi-pythagorean fields

In this section, we state some results on valuations on quasi-pythagorean fields. By a field F, we shall always mean a field of characteristic different from two. We denote by $\dot{F}$ the multiplicative group of F. Let v be a valuation on F. The value group Γ will always be written multiplicatively. The objects: the valuation ring of v, the maximal ideal of v, the group of units and the residue class field of v will be denoted by A, M, U and $\bar{F}$ respectively. For a subset $B\subseteq A$, we put $\bar{B}=\{x+M\in\bar{F} | x\in B\}$.

We write v' for the composition $\dot{F}\xrightarrow{v}\Gamma\to\Gamma/\Gamma^2$. For simplicity, we also write v' for the induced homomorphism $\dot{F}/\dot{F}^2\to\Gamma/\Gamma^2$. There is a natural short exact sequence

$$1 \longrightarrow U\dot{F}^2/\dot{F}^2 \longrightarrow \dot{F}/\dot{F}^2 \xrightarrow{v'} \Gamma/\Gamma^2 \longrightarrow 1 .$$

Since the three groups involved are all elementary 2-groups, this is a split exact sequence. We shall choose and fix a splitting λ: $\dot{F}/\dot{F}^2\to U\dot{F}^2/\dot{F}^2$. Composing λ with the natural maps $U\dot{F}^2/\dot{F}^2\cong U/U\cap\dot{F}^2\to(\bar{F})^{\cdot}/(\bar{F})^{\cdot 2}$, we get a surjective homomorphism λ': $\dot{F}/\dot{F}^2\to(\bar{F})^{\cdot}/(\bar{F})^{\cdot 2}$. By abuse of notation, the composition of this