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Generalizations of Witt algebras over a field of characteristic zero

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Introduction

In this paper we investigate the structure of generalizations of Witt algebras over a field $\mathfrak{k}$ of characteristic zero, and consider a class of infinite-dimensional simple Lie algebras over $\mathfrak{k}$. Let I be a non-empty index set and G be an additive subgroup of $\prod_{i\in I}\mathfrak{k}_i^+$, where $\mathfrak{k}_i^+$ $(i \in I)$ are copies of the additive group $\mathfrak{k}$. Let $W(G, I)$ be the Lie algebra over $\mathfrak{k}$ with basis $\{w(a, i) \mid a \in G, i \in I\}$ and the multiplication

$$[w(a, i), w(b, j)] = a_j w(a+b, i) - b_i w(a+b, j),$$

where $i, j \in I$ and $a=(a_i)_{i\in I}$, $b=(b_i)_{i\in I} \in G$. The Lie algebra $W(G, I)$ is infinite-dimensional if $G \neq 0$.

We note that if the field $\mathfrak{k}$ is of characteristic $p>0$, then $W(G, I)$ is isomorphic to the generalized Witt algebra defined by Kaplansky [3]. It is known that the generalized Witt algebra is simple if G is "total" and $\mathfrak{k}$ is of characteristic $p>2$ [3] (see also Ree [5], Seligman [6], and Wilson [7]). It is also known that $W(G, I)$ is simple if $|I|=1$, $G\neq 0$, and $\mathfrak{k}$ is of characteristic $\neq 2$ [2, p. 206].

The main results of this paper are as follows: If $G\neq 0$, then $W(G, I)$ is a direct sum of the unique maximal ideal R of $W(G, I)$ and a simple subalgebra S of $W(G, I)$, where S is isomorphic to $W(H, J)$ for some H and J (Theorem 3.1). If $G\neq 0$, then the following statements are equivalent: (i) $W(G, I)$ is simple; (ii) $R=0$; (iii) the center of $W(G, I)$ is 0; (iv) G is "total" (Corollary 3.2). $W(G, I)$ is a finitely generated Lie algebra if and only if I is a finite set and G is a finitely generated group (Theorem 4.1). If $I=\{1,\dots, n\}$ and $G=\oplus_{i=1}^n \mathbf{Z}_i$, then $W(G, I)$ is isomorphic to the derivation algebra of $\mathfrak{k}[x_1, x_1^{-1},\dots, x_n, x_n^{-1}]$ (Proposition 4.2).

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1. Notation and preliminary results

Throughout this paper the ground field $\mathfrak{k}$ is of characteristic zero and Lie