HIROSHIMA MATH. J.
16 (1986), 665–669

On the (non)compactness of the radial sobolev spaces

Yukiyoshi EBIHARA and Tomas P. SCHONBEK

(Received February 25, 1986)

1. Introduction and basic compactness results

We say that a normed space V is embedded in a normed space H, and write $V \hookrightarrow H$, if V is a linear subspace of H and the injection mapping $x \to x$ from V to H is continuous. If, in addition, the injection mapping is a compact operator from V to H, we say that V is compactly embedded in H and write $V \Subset H$. From the well known Sobolev lemma, we know that

$$(1) \qquad H^1(\boldsymbol{R}^n) \hookrightarrow L^q(\boldsymbol{R}^n)$$

holds for $2 \leq q \leq 2n/(n-2)$ if $n > 2$. Moreover, if Ω is a bounded piece-wise smooth domain in $\boldsymbol{R}^n$, then $H^1(\Omega)$ is compactly embedded in $L^q(\Omega)$ if $2 \leq q < 2n/(n-2)$. However, this fails to hold if Ω is not bounded. To see, for example, that we cannot replace the embedding in (1) by a compact embedding, let $\phi_m(x) = \phi(x + me)$ $(m = 1, 2, \ldots)$, where ϕ is a non-zero element of $C_0^\infty(\boldsymbol{R}^n)$ and e is a unit vector in $\boldsymbol{R}^n$. The sequence $\{\phi_m\}$ is clearly bounded in $H^1(\boldsymbol{R}^n)$ but does not have a subsequence converging in L^q.

When solving differential equations, we sometimes construct approximate solutions in a suitable framework of function spaces and discuss their convergence. For non-linear equations specially, the best way to prove convergence in some space is to prove that the sequence of approximate solutions is bounded in some space which is compactly embedded in the original space. There is a lot of information available concerning compact and non-compact embeddings in the theory of Sobolev spaces (cf. [1], [2], [3], [4]). In this paper we restrict our attention to the spaces

$$H_r^m(\boldsymbol{R}^n) := \{u \in H^m(\boldsymbol{R}^n) : u = u(|x|)\};$$

i.e., the class of all radial functions $u(|x|)$ whose derivatives of order up to m belong to $L^2(\boldsymbol{R}^n)$ $(m = 0, 1, 2, \ldots)$; $n \geq 3$. We denote by $\|\cdot\|$ the L^2-norm in $\boldsymbol{R}^n$ and we set for $m \geq 0$

$$\|u\|_m^2 = \|u\|^2 + \| |\xi|^m \hat{u} \|^2,$$

where $\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix \cdot \xi} u(x) dx$ is the Fourier transform of u. The norm $\|\cdot\|_m$ is equivalent to the standard norm of $H^m(\boldsymbol{R}^n)$; we employ this norm in $H_r^m(\boldsymbol{R}^n)$. We have: