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Immersions of real projective spaces into complex projective spaces

Dedicated to Professor Masahiro Sugawara on his 60th birthday

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§1. Introduction

Given two differentiable manifolds M, N and a continuous map $f: M \to N$, we denote by $I[M, N]_f$ the set of regular homotopy classes of immersions of M into N, which are homotopic to f, and by $I[M, N]$ the set of all regular homotopy classes of immersions of M into N.

As for the set $I[M, N]$, some results have so far been obtained when N is not Euclidean space. For example, for the existence of immersions of $P^n(Q)$ into $P^m(C)$ or $P^m(R)$, of $P^n(C)$ into $P^m(C)$, of $P^n(R)$ into $P^m(R)$, and of $L^n(p)$ into $L^m(p)$, see [1], [4] and [15], [8], and [5], respectively, and for the classification of immersions of $P^n(R)$ into $P^m(R)$, see [7]–[9], where $P^k(F)$ is the F-projective space of F-dimension k and $L^k(p)$ is the lens space mod p. But the above results are smaller in number than those when N is Euclidean space.

In this article we shall study the set $I[P^n(R), P^m(C)]$ and $I[P^n(R), P^m(C)]_f$ for any map $f: P^n(R) \to P^m(C)$ $(n \leqq 2m-1)$.

Here we note the fact that $[P^n(R), P^m(C)] = Z_2$ if $n \leqq 2m$ (see (2.5) below).

Let $i: P^n(R) \to P^n(C) \subset P^m(C)$ $(n \leqq m)$ be the natural embedding defined by regarding real numbers as complex numbers and let $c: P^n(R) \to P^m(C)$ be a constant map. Then we shall prove the following theorems:

THEOREM A. *Assume that $n \geqq 2$. Then*

(i) *for $n \leqq m$, the natural embedding i is not null-homotopic,*

(ii) *for $n > m$, any immersion of $P^n(R)$ into $P^m(C)$, if any, is always null-homotopic.*

THEOREM B. *Assume that $n > 2$. Then*

(i) *both for $f = i$ and $f = c$,*

$$I[P^n(R), P^n(C)]_f = \begin{cases} Z & n \equiv 0 \mod 2, \\ Z_2 & n \equiv 1 \mod 2; \end{cases}$$

(ii) *if $m < n < 2m-1$, then*