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Generic solvability of the equations of Navier-Stokes

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1. Introduction

Let $\Omega \subset \mathbf{R}^3$ be a bounded domain in $\mathbf{R}^3$ with a smooth boundary $\partial\Omega$; $\partial\Omega$ is of class C^∞. We consider the equations of Navier-Stokes

$$(1.1) \qquad u' - \Delta u + u \cdot \nabla u + \nabla \pi = f, \quad \operatorname{div} u = 0, \quad u|_{\partial\Omega} = 0, \quad u(0) = u_0$$

on the cylindrical domain $\Omega \times (0, T) \subset \mathbf{R}^4$ with some $T > 0$, and we investigate strong solutions u of (1.1); these are solutions with $u \in L^p(0, T; H^{2,p}(\Omega)^3 \cap \mathring{H}^{1,p}(\Omega)^3)$ and $u' \in L^p(0, T; L^p(\Omega)^3)$ for some p with $2 \leqq p < \infty$.

Using the projection P_p: $L^p(\Omega)^3 \to H_p(\Omega)$ from $L^p(\Omega)^3$ onto the subspace $H_p(\Omega) \subset L^p(\Omega)^3$ of divergence free functions with zero normal component on $\partial\Omega$ (in the sense of [3]), we can write (1.1) in the following equivalent form as an evolution equation in $H_p(\Omega)$:

$$(1.2) \qquad u' + A_p u + P_p(u \cdot \nabla u) = P_p f, \quad u(0) = u_0, \quad 0 \leqq t \leqq T.$$

Here A_p: $v \to A_p v$: $= -P_p \Delta v$ denotes the Stokes operator with domain $D(A_p) := H^{2,p}(\Omega)^3 \cap \mathring{H}^{1,p}(\Omega)^3 \cap H_p(\Omega)$. We can define the fractional powers A_p^α of A_p with $0 \leqq \alpha \leqq 1$ and domain $D(A_p^\alpha) \supset D(A_p)$ as in [6]. Let $f \in L^p(0, T; L^p(\Omega)^3)$ and $u_0 \in D(A_p^{1-(1/p)+\delta})$ with some δ, $0 < \delta < 1/p$ (take $u_0 \in D(A_p)$ for example). Then a strong solution u of (1.1) or (1.2) is defined by the conditions $u \in L^p(0, T; D(A_p))$, $u' \in L^p(0, T; L^p(\Omega)^3)$ and (1.2).

The existence of strong solutions of (1.1) for arbitrary $T > 0$ is an important unsolved problem up to now. Therefore it is interesting to know properties of the set

$$R(u_0) := \{f \in L^p(0, T; L^p(\Omega)^3) \mid (1.2) \text{ has a unique strong solution } u \text{ with data } f, u_0\}$$

for a fixed initial value $u_0 \in D(A_p^{1-(1/p)+\delta})$. It is not known whether or not $R(u_0) = L^p(0, T; L^p(\Omega)^3)$; however we can prove some density properties of this set. This gives us some information how many f do exist such that (1.1) is strongly solvable.

Solonnikov's theory of local solvability [10; §10] tells us that $R(u_0) \subset L^p(0, T; L^p(\Omega)^3)$ is an open subset. In case $p = 2$ it has been shown that $R(u_0)$