HIROSHIMA MATH. J.
17 (1987), 471–487

An elementary proof of the Trombi theorem for the Fourier transform of $\mathscr{C}^p(G: F)$

Masaaki EGUCHI and Masato WAKAYAMA

(Received January 19, 1987)

§1. Introduction

Let G and $\mathfrak{g}$ be a real connected noncompact semisimple Lie group with finite center and its Lie algebra respectively. Let $G=KAN$ be an Iwasawa decomposition of G and $\mathfrak{g}=\mathfrak{k}+\mathfrak{a}+\mathfrak{n}$ the corresponding decomposition of $\mathfrak{g}$. Denote by $\hat{K}$ the set of all equivalence classes of irreducible unitary representations of K. Let $\boldsymbol{F}\subset\hat{K}$, $|\boldsymbol{F}|<\infty$ and $0<p\leq 2$. Let $\mathscr{C}^p(G: \boldsymbol{F})$ be the L^p Schwartz space on G of type $\boldsymbol{F}$. It follows from the definition that if $0<p'<p\leq 2$ then

$$C_c^\infty(G: \boldsymbol{F}) \subset \mathscr{C}^{p'}(G: \boldsymbol{F}) \subset \mathscr{C}^p(G: \boldsymbol{F}) \subset \mathscr{C}^2(G: \boldsymbol{F}) = \mathscr{C}(G: \boldsymbol{F}).$$

The images of $\mathscr{C}^p(G: \boldsymbol{F})$ by the Fourier transform are characterized by Harish-Chandra [9(c, d, e)] for $p=2$ and general rank cases, and by Trombi [12(c)] for $0<p<2$ and $\mathrm{rk}\,(G/K)=1$ case, respectively. One of the most difficult parts of the theory in [12(c)] is to show the continuity of the inverse Fourier transform. To prove the main theorem in [12(c)], Trombi [12(b)] investigated the asymptotic behavior of the Eisenstein integral at infinity. He gave, taking some terms of the Harish-Chandra expansion of the spherical function as an approximation for it, a uniform estimate for the difference between them for $\nu\in F$ apart from a compact set including the origin, where F denotes $(-1)^{1/2}\mathfrak{a}^*$ ($\mathfrak{a}^*$ the real dual space of $\mathfrak{a}$). But the use of the approximation, instead of the whole series expansion of the spherical function, and the exclusion of a compact set in the approximation theorem, made the proof of the continuity of the Fourier inverse map rather complicated.

On the other hand, Eguchi-Hashizume-Koizumi [4] obtained the Gangolli estimates for the coefficients of the Harish-Chandra expansions of Eisenstein integrals. Our purpose of this paper is to show that we can give an elementary proof of the continuity of the wave packets, the Fourier inverse map, by using the whole expansion and the Gangolli estimates. But unfortunately, our proof cannot remove the K finite condition on $\mathscr{C}^p$ functions (see Remark in Section 6).

In Section 3, we review the Harish-Chandra expansion of the Eisenstein integral and the Gangolli estimates for its coefficients. To explain the instruments which we use in Section 6, we recall in Sections 4 and 5, the notion of the Fourier transform of $\mathscr{C}^p(G: \boldsymbol{F})$ from [12(c)]. We give in Section 6 an elementary proof of the continuity of the wave packets.