HIROSHIMA MATH. J.
18 (1988), 69–72

Convergence of sum product of a martingale difference sequence

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(Received May 13, 1987)

1. Introduction

Let $\{x_k(t)\}_{k\in N}$ (N is the collection of all natural numbers) be a sequence of complex functions on [0, 1] such that $x_k(t)+1\neq 0$ for every $t\in[0, 1]$. Then the convergence of the product $\prod_k(1+x_k(t))$ has been investigated by many authors in connection with the convergence of the sum $\sum_k x_k(t)$. For example G. H. Hardy [1] proved that if $\{a_k\}$ is a sequence of positive numbers which converges monotonically to zero and $\sum_k a_k^n$ diverges for every $n\in N$, then $\prod_k(1+a_k e^{2\pi ikt})$ diverges for every rational number t. J. E. Littlewood [2] proved that if $\{a_k\}$ is a sequence of positive numbers converges monotonically to zero, then $\prod_k(1+a_k e^{2\pi ikt})$ converges for every irrational number t with possible exception of the Liouville numbers. In the measure theoretical point of view, L. Carleson's theorem implies that if $\sum_k |a_k|^2<+\infty$, then $\prod_k(1+a_k e^{2\pi ikt})$ converges almost surely. All of these discussions concerned the convergence or the divergence of $\sum_k a_k e^{2\pi ikt}$.

The author investigated this problem from the probabilistic point of view and proved in [4] that if $\{X_k\}$ is a sequence of independent random variables with mean zero such that $1+X_k>0$, a.s., for every k, then the almost sure convergence of $\prod_k(1+X_k)$ is equivalent to that of $\sum_k X_k$. In this paper we shall extend this result to a martingale difference sequence and prove the following theorem.

THEOREM 1. *Let $\{X_k, \mathscr{B}_k\}$ be a martingale difference sequence such that $X_k+1>0$, a.s., for every k. Then $\prod_k(1+X_k)$ converges almost surely if and only if $\sum_k X_k$ converges almost surely.*

As an application we shall give a new criterion for the absolute continuity of locally equivalent measures.

2. Proof of Theorem 1

A sequence of random variables $\{X_k\}$ is a *submartingale difference sequence* iff X_k is $\mathscr{B}_k$-measurable and

$$E[X_{k+1}\,|\,\mathscr{B}_k]\geq 0,\quad \text{a.s.},$$