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Affine semigroups on Banach spaces

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This note treats strongly continuous one-parameter affine semigroups on a Banach space X. An affine semigroup decomposes into a linear part and a translation part. These parts are reassembled here as an "augmented" linear semigroup in one higher dimension, and the latter is used to characterize the affine semigroup. Relations among the infinitesimal generators (i.g.) of an affine semigroup, its linear part, and its augmented semigroup are studied. It is shown that the translation part is completely determined from the linear part by an element of $(X \times X)/G(U)$, where $G(U)$ is the graph of the i.g. of the linear part. Also obtained are necessary and sufficient conditions for a curve to be the translation part of some affine semigroup. An application of these conditions is the so-called "screw line" studied by von Neumann and Schoenberg.

Affine concepts are an almost trivial modification of linear ones. The treatment here is intended to aid in the discovery of the correct nonlinear generalizations of familiar linear concepts. Thus affine versions of one-parameter groups, compact semigroups, and analytic semigroups are studied, along with affine cosine functions.

§ 1. Affine semigroups and associated linear semigroups

Except where otherwise noted, Banach spaces are taken to be real.

By a strongly continuous one-parameter affine semigroup on a Banach space X, or *affine semigroup* for short, is meant a one-parameter family $\{S(t): t \geq 0\}$ of continuous affine transformations on X with the properties

(s_1) $S(0)=I$ (the identity operator on X), $S(t+s)=S(t) \circ S(s)$ for s, $t \geq 0$; and
(s_2) for each x in X, $t \mapsto S(t)x$ is a continuous function from $[0, \infty)$ into X.

A *linear semigroup* is defined similarly except that "linear" transformations replace "affine" transformations. Strong continuity for $t \geq 0$ will be assumed in both the linear and affine cases.

Affine semigroups arise naturally in the study of linear differential equations with a nonhomogeneous (constant) term. Applications of the affine theory

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