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Maximal extensions of ordered fields

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The notion of maximal ordered fields was first introduced in [2] and [3] by the authors. The existence and the uniqueness for a given rank were mainly discussed there.

In this paper, we say that K is a *maximal extension* of an ordered field F if $\psi_{K/F}$ is bijective and K is a maximal ordered field. The aim of this paper is to develope serveral basic properties of maximal extensions. Namely, for maximal extensions K_i/F_i $(i=1, 2)$ and a given isomorphism σ: $F_1 \simeq F_2$ as ordered fields, there exists an extension σ': $K_1 \simeq K_2$ of σ; we also show that there can be infinitely many such extensions. Moreover we show that, for any extension K/F_1 such that K is a maximal ordered field, there exists an F_1-embedding $K_1 \to K$.

§1. Maximal extensions

For an ordered field F, $A_0 := A(F, \mathrm{Q}) = \{a \in F; |a| < b \text{ for some } b \in \mathrm{Q}\}$ is the finest valuation ring, that is, every convex valuation ring of F is a localization of A_0 and conversely. The set $\mathscr{C}(F)$ of all convex valuation rings of F is a totally ordered set under the inclusion relation. Let G_0 be the value group of the finest valuation defined by A_0. It is clear that $\mathscr{C}(F)$ is isomorphic to the set $\mathscr{H}(F)$ of all convex subgroups of G_0 as totally ordered sets. If K/F is an extension of ordered fields, we have a surjection $\psi_{K/F}$: $\mathscr{C}(K) \to \mathscr{C}(F)$ defined by $\psi(B_i) = B_i \cap F$, $B_i \in \mathscr{C}(K)$ (cf. [2], §1). A pair (A, B) of subsets of F is called a cut of F if $A \cup B = F$ and $a < b$ for any $a \in A$ and $b \in B$, where A or B may be an empty set. It follows from [1], Theorem 1.2, that if F is real closed, then there is a one to one correspondence between the set of all cuts of F and the set of all orderings of $F(x)$, where $F(x)/F$ is a simple transcendental extension. For a subset C of F and an element a of F, we write $C < a$ if $c < a$ for any $c \in C$. We say that K is a maximal ordered field if $\psi_{L/K}$ is not bijective for any proper extension L/K of ordered fields (cf. [3], Definition 2.1).

DEFINITION 1.1. For an ordered field F, let K/F be an extension of ordered fields. When K is a maximal ordered field and $\psi_{K/F}$ is bijective, we say that K is a *maximal extension* of F.

For any ordered field F, there exists a maximal extension of F by [3], Theorem 3.3.

LEMMA 1.2. *Let F be a real closed field and K be a maximal extension of F. Let*