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Reductions of graded rings and pseudo-flat graded modules

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Introduction

The aim of this paper is to develop a theory of *reductions* for graded rings and to study graded modules using this theory. In particular, we introduce a certain class of graded modules which we call *pseudo-flat* graded modules and examine some of their properties. Our theory of reductions of graded rings is a natural generalization of the theory of reductions of ideals due to Northcott and Rees [16], and the techniques used are similar to the ones in the case of ideals. But the viewpoint of general graded rings greatly clarifies the situations and is useful even in the case of ideals.

In §1 and §2, we define the *analytic spread* and the pseudo-flatness of graded modules, and prove some elementary facts about them.

In §3, we introduce the notion of reductions of homogeneous graded rings with respect to finitely generated graded modules. Then we prove a fundamental theorem in the theory of reductions, namely, the existence of minimal reductions and the characterization of minimal reductions by the analytic spread (cf. Theorem 3.3). By this theorem, we can give the structure theorem for pseudo-flat graded modules (cf. Theorem 3.4).

In §4, using this structure theorem, we examine some properties of pseudo-flat graded modules.

In §5, making use of minimal reductions, we introduce a numerical invariant of a graded module which we call the *reduction exponent*, and study properties of graded modules by this invariant. Especially, we compare the reduction exponent with *Castelnuovo's regularity* which the author introduced in [17].

Notation and terminology: Throughout this paper, all rings are commutative noetherian rings. Any graded ring $A=\oplus_{n\in Z}A_n$ is positively graded (i.e., $A_n=0$ for all $n<0$), and is generated over $A_0=R$ by elements of degree one. Then we say that A is a *homogeneous* R-algebra. We put $A_+=\oplus_{n>0}A_n$. Let R be a ring, I an ideal of R and E an R-module. $\mathrm{Min}_R(E)$ denotes the set of minimal elements in $\mathrm{Supp}_R(E)$. $\mu(E)$ denotes the smallest number of generators of E. For a homogeneous R-algebra A, put $\mathrm{emb}(A)=\mu(A_1)$ (the *embedding dimension* of A). If A is a homogeneous algebra over a field and M is a finitely generated graded A-module, then $\mathrm{e}(M)$ denotes the multiplicity of M. We put $R(I, E)=\oplus_{n\geqq 0}I^nE$, $G(I, E)=\oplus_{n\geqq 0}I^nE/I^{n+1}E$, $R(I)=R(I,$