HIROSHIMA MATH. J.
19 (1989), 333–346

Modularity conditions in Lie algebras

Kevin BOWMAN and David TOWERS

(Received November 25, 1987)
(Revised July 25, 1988)

1. Introduction

For several years some authors have been studying the properties of algebras by considering the subalgebra lattice structure. Throughout this paper lattice conditions will be defined in an algebra context. It will become clear that much of the earlier work will hold true for any lattice.

A subalgebra U of an algebra A is called *modular* in A if it is a modular element in the lattice of subalgebras of A; that is

$$\langle U, B \rangle \cap C = \langle B, U \cap C \rangle \quad \text{for all subalgebras } B \subseteq C$$

and

$$\langle U, B \rangle \cap C = \langle B \cap C, U \rangle \quad \text{for all subalgebras } U \subseteq C.$$

(Here $\langle X, Y \rangle$ denotes the subalgebra of A generated by X and Y.) We say A is *completely modular* if every subalgebra is modular in A.

Modular subalgebras were studied by Amayo and Schwarz in [1]. A natural question to ask is, "Can the hypothesis of modularity be weakened in such a way that useful information about the algebra can still be obtained?" The answer to this question is "yes", but it is unclear as to what is the "best" weakened hypothesis as there are several sensible versions and their relationships to one another are still unclear. It is to a better clarification of this situation and a deeper understanding of different types of modularity that this paper is directed.

Throughout this section U will denote a subalgebra of a general algebra A. We shall denote that U is maximal in B by $U < B$.

We now give some of the key definitions: We say that U is *upper semi-modular* in A or *u.s.m.* in A if for every subalgebra B of A such that $U \cap B < U, B$ then $U, B < \langle U, B \rangle$. We say that A is *completely upper semi-modular* or *completely u.s.m* if every subalgebra of A is u.s.m. in A. We say that U is *lower semi-modular* in A or *l.s.m.* in A if for every subalgebra B of A such that $U, B < \langle U, B \rangle$ then $U \cap B < U, B$. We say that A is *completely lower semi-modular* or *completely l.s.m.* if every subalgebra of A is l.s.m. in A. (Note that completely u.s.m. and completely l.s.m. algebras were called upper semi-modular and lower semi-modular respectively by Kolman, Gein and Varea.)