Three Riemannian metrics on the moduli space
of 1-instantons over $CP^2$

Katsuhiro Kobayashi

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1. Introduction

The natural metric on 5-sphere of radius 1 induces the Fubini-Study metric $g_{FS}$ on the complex projective plane $CP^2$. The moduli space $\mathcal{M}$ of 1-instantons over $(CP^2, g_{FS})$ is homeomorphic to the cone on $CP^2$ (Buchdahl [B] and Furuta [F]). The generic part $\mathcal{M}^*$ of the moduli space carries three natural Riemannian metrics $g_I$ ($I = I, II$ and $I - II$). We refer to Matumoto [M] for the definition of the Riemannian symmetric tensors. In this paper we will give explicit formulas of the metrics and study their basic geometric properties.

Buchdahl and Furuta defined an $SU(3)$-equivariant diffeomorphism $F$: $CP^2 \times (0, 1) \cong \mathcal{M}^* \cong \mathcal{M}$-{cone point}. We use a local coordinate system $C^2 \times (0, 1) \rightarrow CP^2 \times (0, 1)$ defined by $(W_1, W_2, \lambda) \rightarrow ([1, W_1, W_2], \lambda)$ with $W_1 = X_1 + iX_2$ and $W_2 = X_3 + iX_4$. Note that $F(C^2 \times (0, 1))$ is open and dense in $\mathcal{M}^*$. The metric tensors split with respect to this coordinate system as

$$F^*g_I = \varphi_I(\lambda) d\lambda^2 + \psi_I(\lambda) g_{FS} \quad (I = I, II \text{ and } I - II).$$

More explicitly, we can write $\varphi_I(\lambda)$ and $\psi_I(\lambda)$ by using a new parameter $Z = 1 - \lambda^2$ as follows:

$$\varphi_I(\lambda) = 8\pi^2(Z^2 \log Z + 3Z \log Z - 3Z^2 + 2Z + 1)/(1 - Z)^3,$$

$$\psi_I(\lambda) = 4\pi^2(-6Z^2 \log Z + Z^3 + 6Z^2 - 9Z + 2)/(1 - Z)^2;$$

$$\varphi_{II}(\lambda) = 16\pi^2(Z^2 - 2Z + 6)/15Z^2, \quad \psi_{II}(\lambda) = 8\pi^2(-3Z^2 - 4Z + 12)(1 - Z)/15Z;$$

$$\varphi_{I-II}(\lambda) = \varphi_{II}(\lambda), \quad \psi_{I-II}(\lambda) = 24\pi^2(Z^4 - Z^3 + 2Z^2 + 8)(1 - Z)/5Z(Z + 2)^2.$$

In fact, $\varphi_I(\lambda)$ and $\psi_I(\lambda)$ are positive for $0 < \lambda < 1$ and $g_I$ defines actually the positive definite Riemannian metrics for not only $J = I$ but also $J = II$ and $I - II$.

From the above formulas or their asymptotic ones given in §4, we get the following proposition, where $K_J(u, v)$ $(J = I, II \text{ and } I - II)$ denote the sectional curvatures of $F^*g_I$.

**Proposition.** (a) As $\lambda \to 1$ (near the collar) all the sectional curvatures converge to the negative constant $-5/32\pi^2$ for $(\mathcal{M}^*, g_{II})$ and $(\mathcal{M}^*, g_{I-II})$. On $(\mathcal{M}^*, g_I)$, we can induce a $C^\infty$ metric on $\partial \mathcal{M}$ so that $(\partial \mathcal{M}, g_I)$ is isometric to