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On some contractive properties for the heat equations

Tetsuya KOYAMA

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0. Introduction

This work is concerned with contractive properties for the solutions of the following initial boundary value problem (IBVP) for the heat equation:

$$\text{(IBVP)} \qquad \begin{cases} \dfrac{\partial}{\partial t} u(x, t) = \Delta u(x, t)\,, & x \in \Omega\,,\ t > 0\,, \\ u(x, 0) = u_0\,, & x \in \Omega\,, \\ u(x, t) = 0\,, & x \in \partial\Omega\,,\ t > 0\,, \end{cases}$$

where Ω is a bounded domain in $\mathbf{R}^N$ and $\partial\Omega$ denotes its boundary.

For a solution $u(x, t)$ of (IBVP), consider the following type of contraction property:

$$(\mathrm{D}_p) \qquad \|\nabla u(\cdot, t)\|_{L^p(\Omega)} \leq \|\nabla u(\cdot, s)\|_{L^p(\Omega)}\,, \qquad 0 < s \leq t\,,$$

for $1 \leq p \leq \infty$. Here, ∇u is the gradient of u. In [1], H. Engler showed that (D_p) holds for any domain Ω, if p is close to 2 in some sense. It is well known that (D_2) holds for any domain because $\|\nabla u(\cdot, t)\|_{L^2(\Omega)}$ is the Dirichlet integral of $u(\cdot, t)$. Furthermore, if the mean curvature H of $\partial\Omega$ is nonnegative (in this case, Ω is said to be H-convex), it is known that (D_p) holds for any p. (See [1].) Engler generalized this result to the case of arbitrary domains. In this note, we consider three functionals OSC_ε, H_α and Lip which are equivalent to the functional

$$u \mapsto \max\{|u(x) - u(y)|\,|\,x, y \in \bar{\Omega}, |x - y| \leq \varepsilon\}\,,$$

the usual Hölder norm and Lipschitz norm, respectively. These functionals, as well as the functional $u \mapsto \|\nabla u\|_{L^p(\Omega)}$, represent the regularity of u. The aim of this note is to show that the above three functionals have the same type of contractive properties as in (D_p) under the assumption that Ω is convex.

1. Three kinds of Lyapunov functionals for Δ

Let Ω be a bounded convex domain in $\mathbf{R}^N$. In what follows, we consider the Banach space