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Geometry of minimum contrast

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1. Introduction

Such concepts as information, entropy, divergence, energy and so on play an important role in mathematical sciences to research random phenomena. This paper tries a unified approach to measurement of these notions, in particular the geometrical structure induced by a contrast function. In the mathematical formulation a contrast function ρ on a manifold M is defined by the first requirement for distance: $\rho(x, y) \geq 0$ with equality if and only if $x = y$, see Eguchi [2] for various examples. A simple example is found in

$$\rho_1(\boldsymbol{p}, \boldsymbol{q}) = \sum_{i=1}^{n+1} p_i(\log p_i - \log q_i)$$

on the n-simplex $\mathscr{S} = \{\boldsymbol{p} = (p_1, \ldots, p_{n+1})\colon \sum_{i=1}^{n+1} p_i = 1, 0 < p_i < 1\}$. This function is called the Kullback information in the context that $\boldsymbol{p}$ and $\boldsymbol{q}$ are the vectors of probabilities for $n + 1$ disjoint events, see [2] for other examples and construction for ρ. Thus a contrast function is generally not assumed to be symmetric as seen in ρ_1.

We discuss on the manifold M instead of $\mathscr{S}$ on the assumption of finite dimensionality because we wish to investigate contrast functions or functionals over not only $\mathscr{S}$ but also a general space of probability measures. A new geometry on M by means of ρ is presented: a Riemannian g, a pair (∇, ∇^*) of torsion-free connections and a pair (D, D^*) of second-order differentials. The asymmetry of ρ leads to different two connections ∇ and ∇^* such that $1/2\ (\nabla + \nabla^*)$ is the Riemannian connection. Lauritzen [3] calls (M, g, T) a statistical manifold, where T is the third order tensor representing the difference between ∇ and ∇^*. In general such a pair (∇, ∇^*) is called conjugate in the sense that if M is curvature-free with respect to ∇, then M is also curvature-free with respect to ∇^*. Nagaoka and Amari [6] extended a notion of locally Euclidean space: If M is curvature-free with respect to ∇, then there exists a pair of local coordinates (x^i, U) and (x_i^*, V) such that

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_j^*}\right) = \delta_i{}^j \qquad \text{(Kronecker's delta)}$$

on $U \cap V$. In Section 2 we present a further conjugacy property introduced