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Continuity properties of potentials and Beppo-Levi-Deny functions

Dedicated to Professor M. Ohtsuka on the occasion of his seventieth birthday

Yoshihiro MIZUTA

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1. Introduction

In this paper we first study the behavior of Riesz potentials of functions near a given point, which may be assumed, without loss of generality, to be the origin. For $0 < \alpha < n$ and a nonnegative measurable function f on R^n, we define $U_\alpha f$ by

$$U_\alpha f(x) = \int_{R^n} |x - y|^{\alpha - n} f(y)\, dy.$$

It is easy to see that $U_\alpha f \not\equiv \infty$ if and only if

$$(1.1) \qquad \int_{R^n} (1 + |y|)^{\alpha - n} f(y)\, dy < \infty.$$

By Sobolev's imbedding theorem, we know that if f is a nonnegative function in $L^p(R^n)$ satisfying (1.1), and if $\alpha p > n$, then $U_\alpha f$ is continuous at the origin (in fact, on R^n); however, in case $\alpha p \leq n$, $U_\alpha f$ may fail to be continuous at the origin. Thus, our main concern in this paper is the bordering case $p = n/\alpha$, and one of our aims is to find a condition on f, which is stronger than the condition that $f \in L^p(R^n)$ with $p = n/\alpha$ but assures the continuity at 0 of $U_\alpha f$.

For this purpose, we assume that f satisfies a condition of the form:

$$(1.2) \qquad \int_{R^n} \Phi_p(f(y))\, \omega(|y|)\, dy < \infty.$$

Here $\Phi_p(r)$ and $\omega(r)$ are positive monotone functions on the interval $(0, \infty)$ with the following properties:

($\varphi 1$) $\Phi_p(r)$ is of the form $r^p \varphi(r)$, where $1 \leq p < \infty$ and φ is a positive nondecreasing function on the interval $[0, \infty)$.

($\varphi 2$) φ is of logarithmic type, that is, there exists $A_1 > 0$ such that