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Any statistical manifold has a contrast function — On the C^3-functions taking the minimum at the diagonal of the product manifold

Dedicated to Professor Masahisa Adachi on his 60th birthday

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§1. Introduction

A statistical manifold is defined by Lauritzen [5] and nothing but a Riemannian manifold (M, g) with a symmetric covariant tensor T of order 3. The symmetric tensor defines a pair of torsion free dual connections and vice versa; The latter geometry has been studied by Amari and others in connection with statistical inferences (cf. [1] and [2]). Moreover, Eguchi [3] has proved that a contrast function also gives all the data of the statistical manifold if it exists; We will find in this paper a contrast function which induces a given statistical manifold.

DEFINITION. A contrast function of a differentiable manifold M is a real-valued smooth function ρ on $M \times M$ such that $\rho(x, y) \geq 0$ with equality if and only if $x = y$ and

$$X_x X_x \rho(x, y)_{|x=y} > 0$$

for any smooth vector field X on M which is non-zero at x.

We use the following notation for a function on M defined by the value of the partial derivative in $M \times M$ with respect to the smooth vector fields $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ on M as in [3]:

$$\rho(X_1 \ldots X_n | Y_1 \ldots Y_m)(z)$$
$$= (X_1)_x \ldots (X_n)_x (Y_1)_y \ldots (Y_m)_y \rho(x, y)_{|x=z, y=z}.$$

In particular, the partial derivative $Z\rho(X_1 \ldots X_n | Y_1 \ldots Y_m)$ on M is equal to $\rho(ZX_1 \ldots X_n | Y_1 \ldots Y_m) + \rho(X_1 \ldots X_n | ZY_1 \ldots Y_m)$.

According to Eguchi [3] the Riemannian metric tensor g is defined by

$$g(X, Y) = -\rho(X | Y).$$

Since the contrast function takes the minimum at the diagonal manifold