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A note on complement ideals of Lie algebras

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Let L be a not necessarily finite-dimensional Lie algebra over any field. In this note we shall give an affirmative answer to Question 1.7 of Aldosray and Stewart [2]: If L is semisimple and I, J are complement ideals of L, then $I \cap J$ always a complement ideal of I? L is called semisimple if L has no non-abelian ideals. Recall that an ideal J of L is complement if there exists an ideal N of L such that $J \cap N = 0$ and $K \cap N \neq 0$ for any ideal K of L with $J \subsetneq K$ [2, p. 5].

THEOREM. *If I and J are complement ideals of a semisimple Lie algebra L, then $I \cap J$ is a complement ideal of I.*

PROOF. By [1, Lemma 2.3] an ideal H of L is a complement ideal of L if and only if H is a centralizer ideal of L, that is, $H = C_L(K)$ for some ideal K of L. Hence I and J are centralizer ideals of L, and then $I \cap J$ is a centralizer ideal of L. So $I \cap J$ is a complement ideal of L as noticed in [2, p. 5]. Then by definition there exists an ideal N of L such that $(I \cap J) \cap N = 0$, and that $K \cap N \neq 0$ for any ideal K of L such that $I \cap J \subsetneq K$. Let $\tilde{N} = I \cap N$. Then $\tilde{N}$ is an ideal of I. Let K be an ideal of I such that $I \cap J \subsetneq K$. We claim that $K \cap \tilde{N} \neq 0$, and to the contrary we assume that $K \cap \tilde{N} = 0$. Then $[K, \tilde{N}] \subseteq K \cap \tilde{N} = 0$. Now let $\langle K \rangle^L = \sum_{n=0}^{\infty} [K, {}_nL]$ be the ideal of L generated by K. Then, since $\tilde{N}$ is an ideal of L, it follows inductively that $[\langle K \rangle^L, \tilde{N}] = 0$. Hence $\langle K \rangle^L \cap \tilde{N}$ is an abelian ideal of L, and $\langle K \rangle^L \cap \tilde{N} = 0$ since L is semisiple. Thus we have $\langle K \rangle^L \cap N = 0$ since $\langle K \rangle^L \subseteq I$. But N is an ideal of L such that $\langle K \rangle^L \cap N \neq 0$ for $\langle K \rangle^L \supsetneq I \cap J$. This is a contradiction, which completes the proof.

References

[1] F. A. M. Aldosray and I. Stewart, Lie algebras with the minimal condition on centralizer ideals, Hiroshima Math. J., **19** (1989), 397–407.

[2] F. A. M. Aldosray and I. Stewart, Ascending chain conditions on special classes of ideals of Lie algebras, Hiroshima Math. J., **22** (1992), 1–13.