HIROSHIMA MATH. J.
24 (1994), 221–241

Lie algebras whose proper subalgebras are either semisimple, abelian or almost-abelian

Vicente R. VAREA*

(Received July 26, 1990)
(Revised August 10, 1992)

Introduction

A number of papers published in recent years have actively studied the relationship between the structure of a Lie algebra to that of the lattice of its subalgebras. In these studies, Lie algebras whose proper subalgebras are either semisimple, abelian or almost-abelian (that we shall call X-algebras for short) have occurred frequently (c.f. [9], [10], [13]). For instance, Lie algebras with a relatively complemented lattice of subalgebras are X-algebras (Gein and Muhin [10]). Of special interest are the Lie algebras in which every subalgebra of dimension > 1 is simple (supersimple Lie algebras).

The purpose of this paper is first to investigate the structure of an X-algebra; and secondly to study upper semi-modular, relatively complemented, supersimple, and minimal non-modular Lie algebras; and thirdly to determine the Lie algebras with a subalgebra lattice of length 3 as well as their corresponding subalgebra lattices.

In section 1, we consider Lie algebras L having an element x such that $C_L(x)$ is abelian and $\dim N_L(C_L(x))/C_L(x) \leq 1$. This class of Lie algebras contains all X-algebras and the Lie algebras having a self-centralizing ad-nilpotent element (which have been determined in [4]). If $N_L(C_L(x)) = C_L(x)$, then we show that x lies in the center of L. If $\dim N_L(C_L(x))/C_L(x) = 1$, then we get that either $N_L(C_L(x))$ is nilpotent, $C_L(x) \lhd L$, or $L/Z(L)$ has a self-centralizing ad-nilpotent element.

Moreover, we prove that the Engel subalgebras of a simple Lie algebra of dimension > 3 are neither almost-abelian nor 3-dimensional simple. This section finishes with two criteria for an element of a Lie algebra to be ad-semisimple.

In section 2, we study the structure of a nonsolvable X-algebra. Solvable X-algebras have been studied in Gein [12, Theorem 3].

In section 3, we use results in the previous sections to study upper semi-modular and relatively complemented Lie algebras. It is known that

* Supported in part by DGICYT #87–54