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A variation formula for harmonic modules and its application to several complex variables

Dedicated to Professor Fumiyuki MAEDA on his 60th birthday

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Introduction

Let R be a compact or noncompact Riemann surface and let γ be a cycle in R. Then there exists a unique square integrable harmonic differential σ in R such that $\int_\gamma \omega = (\omega, *\sigma)_R$ $(= \iint_R \omega \wedge \sigma)$ for all C^2 square integrable closed differentials ω in R. We call σ the reproducing differential for (R, γ). The norm $\lambda = \|\sigma\|_R^2$ is called the harmonic module for (R, γ). L. V. Ahlfors [2] noted their significance in the theory of functions of one complex variable. In this paper we shall show their usefulness in that of several complex variables.

To a complex parameter t in a disk B, we let correspond a covering surface $R(t)$ over the z-plane $\boldsymbol{C}$ with C^∞ smooth boundary $\partial R(t)$ and with branch points $\xi_i(t)$ $(1 \leq i \leq q)$, where q does not depend on $t \in B$. Assume that $\partial R(t)$ varies C^∞ smoothly with the parameter $t \in B$ and that $\xi_i(t)$ is a holomorphic function on B. Thus $\mathscr{R} = \bigcup_{t \in B}(t, R(t))$ is a ramified Riemann domain over $B \times \boldsymbol{C}$. We simply denote $\partial\mathscr{R} = \bigcup_{t \in B}(t, \partial R(t))$, and write $\mathscr{R}: t \to R(t)$, $t \in B$. Now let $\gamma(t)$ be a cycle in $R(t)$ which varies continuously with $t \in B$ in $\mathscr{R}$. As a Riemann surface, each $R(t)$ with $\gamma(t)$ carries the reproducing differential $\sigma(t, \cdot)$ and the harmonic module $\lambda(t)$ for $(R(t), \gamma(t))$. We put $\Omega(t, z) = \sigma(t, z) + i*\sigma(t, z) = f(t, z)dz$ for $z \in R(t)$ and $\|\Omega\|(t, z) = |f(t, z)|$. In [15] and [16] we showed that: *If $\mathscr{R}$ is pseudoconvex over $B \times \boldsymbol{C}$, then* $\dfrac{\partial^2 \lambda(t)}{\partial t \partial \bar{t}} \geq \left\| \dfrac{\partial \Omega}{\partial \bar{t}}(t, \cdot) \right\|_{R(t)}^2$ *for* $t \in B$. *Furthermore, the equality holds for all $t \in B$, if and only if $\mathscr{R}$ is Levi flat.* In this paper, for any $\mathscr{R}: t \to R(t)$, $t \in B$, we shall prove a variation formula for $\lambda(t)$ of the second order, which deduces the above result in the pseudoconvex or Levi flat case. Precisely, let $\varphi(t, z)$ be a C^2 defining function of $\mathscr{R}$, and put, for $(t, z) \in \partial\mathscr{R}$.

$$k_2(t, z) = \left\{ \frac{\partial^2 \varphi}{\partial t \partial \bar{t}} \left| \frac{\partial \varphi}{\partial z} \right|^2 - 2\mathrm{Re} \left\{ \frac{\partial \varphi}{\partial t} \frac{\partial \varphi}{\partial \bar{z}} \frac{\partial^2 \varphi}{\partial \bar{t} \partial z} \right\} + \left| \frac{\partial \varphi}{\partial t} \right|^2 \frac{\partial^2 \varphi}{\partial z \partial \bar{z}} \right\} \Big/ \left| \frac{\partial \varphi}{\partial z} \right|^3$$