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On 4-dimensional closed manifolds with free fundamental groups

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Abstract. Let M be a 4-dimensional connected closed manifold whose fundamental group is a free group of rank m. We will show that the punctured manifold $M - pt$ has the homotopy type of a bouquet $\vee_m S^1 \vee_m S^3 \vee_n S^2$ of spheres for some n.

1. Introduction

Let M be a 4-dimensional connected closed manifold whose fundamental group is a free group $F_m = *_m \mathbf{Z}$ of rank m.

PROPOSITION 1. *$\#_\ell S^2 \times S^2 \# M$ is homeomorphic to $\#_m S^1 \times S^3 \# M_1$ or $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_1$ for some ℓ and some simply connected closed 4-dimensional manifold M_1 according as M is orientable or not. If M has a smooth structure, then the same statement holds for a diffeomorphism.*

With the help of an algebraic argument Proposition 1 would imply

PROPOSITION 2. *The punctured manifold $M - pt$ has the homotopy type of a bouquet $\vee_m S^1 \vee_m S^3 \vee_n S^2$ of spheres for some n.*

Here, we may conjecture that M has the homotopy type of $\#_m S^1 \times S^3 \# M_0$ or $\#_{m-1} S^1 \times S^3 \# S^1 \tilde{\times} S^3 \# M_0$ for some simply connected closed 4-dimensional manifold M_0 according as M is orientable or not. In the case that M is orientable and $m = 1$ this conjecture is true; in fact, Kawauchi [3] proved that M is homeomorphic to $S^1 \times S^3 \# M_0$.

As a corollary of Proposition 2 we have

PROPOSITION 3. *For a connected closed 4-dimensional manifold M the following statements are equivalent: (1) The Lusternik-Schnirelmann category of the punctured manifold $M - pt$* is one. *(2) The fundamental group $\pi_1(M)$ is a free group. (3) The punctured manifold $M - pt$ has the homotopy type of a bouquet of spheres.*

In fact, since $\pi_1(M) = \pi_1(M - pt)$, (1) implies (2) and follows from (3); (2) implies (3) by Proposition 2.

We may ask whether the conditions are equivalent also to the following