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An approach by difference to the porous medium equation with convection

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Introduction

In this paper, we shall discuss the existence and then the uniqueness of the solution to the Cauchy problem for the porous medium equation with "convection" terms:

$$(1)\qquad (\partial/\partial t)u = \Delta\varphi(u) + \sum_{i=1}^{N} F_i(u)_{x_i}, \qquad x\in \boldsymbol{R}^N,\ t>0;$$

$$(2)\qquad u(0, x) = u_0(x), \qquad x\in \boldsymbol{R}^N.$$

Here, $(\cdot)_{x_i}=\partial/\partial x_i$ $(i=1,\cdots,N)$ and $\Delta=\sum_{i=1}^{N}(\partial/\partial x_i)^2$, and φ and F_i $(i=1,\cdots,N)$ are assumed to satisfy the conditions below:

(C1) The function φ is strictly increasing, locally Lipschitz continuous on R^1 and satisfies $\varphi(0)=0$;

(C2) The functions F_i, $i=1,\cdots,N$, are defined on $\boldsymbol{R}^1$, $F_i(0)=0$, and $|F_i(r)-F_i(s)|/|\varphi(r)-\varphi(s)|$ are bounded for r, s in every bounded subinterval of $\boldsymbol{R}^1$.

First, we shall provide a direct method for solving the problem (1)–(2) via the method of difference approximation:

$$\begin{cases} \quad h^{-1}(u(t+h, x) - u(t, x)) \\ = \sum_{i=1}^{N} k^{-2}(T_i(k) - 2I + T_i(-k))\varphi(u(t, x)) \\ \qquad\qquad + \sum_{i=1}^{N}(2k)^{-1}(T_i(k) - T_i(-k))F_i(u(t, x)), \\ T_i(k)u(x) = u(x+ke_i),\ e_i = (0,\cdots,0,\overset{i}{1},0,\cdots,0), \\ i = 1,\cdots,N. \end{cases}$$

We shall explain in Section 1 that this scheme itself converges as $h, k\downarrow 0$ and the limits give rise to a semigroup $\{S(t): t\geqq 0\}$ of contractions on $L^1(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ associated with the problem (1)–(2).

We have once tried an approach similar to that described in the above, to a simpler equation without "convection" terms: