HIROSHIMA MATH. J.
25 (1995), 527–540

On loosely self-similar sets

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(Received May 20, 1994)

1. Introduction

In [7], J. E. Hutchinson set up a theory of strictly self-similar set, which is defined as the unique compact set satisfying the following equality;

$$K = \bigcup_{i=1}^{m} f_i(K)$$

for a given finite set $\{f_i\}_{i=1}^{m}$ of contraction affine maps on a compact subset X of $\mathbb{R}^N$ ($m \geq 2$). Let r_i be the contraction rate of f_i, that is, $|f_i(x) - f_i(y)| = r_i|x - y|$ for $x, y \in X$, $i = 1, 2, \cdots, m$, and let α be the unique solution of $\sum_{i=1}^{m} r_i^\alpha = 1$. In his theory, a Borel probability measure ν on $\mathbb{R}^N$ satisfying $\nu(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(X)) = \prod_{j=1}^{n} r_{i_j}^\alpha$ coincides with the α-dimensional Hausdorff measure on K up to constant, that is, there exists a positive constant C such that $\nu(A) = C\mathrm{H}^\alpha(A)$ for any Borel set $A \subseteq K$. Here H^α denotes the α-dimensional Hausdorff measure.

We now explain his result from the standpoint of Tricot. Tricot [13] showed that for any Borel set $E \subset \mathbb{R}^N$,

$$\text{H-dim}(E) = \sup_{\mu \in \mathscr{M}_E} \{\inf_{x \in E} \phi(\mu; x)\}. \tag{1.1}$$

Where $\mathscr{M}_E = \{\mu$; positive finite Borel measure on $\mathbb{R}^N$ with $\mu(E) > 0\}$ and for $\mu \in \mathscr{M}_E$

$$\phi(\mu; x) = \liminf_{r \downarrow 0} \frac{\log \mu(E \cap B(x, r))}{\log r}. \tag{1.2}$$

H-dim (E) denotes the Hausdorff dimension of E, $B(x, r)$ denotes the closed ball with radius r and center at x. We can easily see that the α-dimensional Hausdorff measure itself attains the supreme in the righthand side of (1.1) in Hutchinson's case. Let

$$K(P_1, P_2, \cdots, P_m) = \{x \in \textstyle\bigcap_{n=1}^{\infty} f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(X);\ \#\{j;\ i_j = k, j \leq n\}/n \to P_k \text{ as } n \to \infty\},$$

$\beta(P_1, P_2, \cdots, P_m)$ denote the Hausdorff dimension of $K(P_1, P_2, \cdots, P_m)$ and $\nu_{(P_1, P_2, \cdots, P_m)}$ be the Borel probability measure satisfying