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On a geometric approach to distributions on a circle

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ABSTRACT. In this paper we aim to discuss a natural measure for means of the distributions on a circle, which plays a role similar to that of a usual mean in a Euclidean space. Because a circle is compact and not flat, it may be noted that we cannot define mean or expectation naturally by the same way as in a Euclidean space.

We introduce a measure of location without embedding the circle into a Euclidean plane. This measure is shown to be an extention of some other measures, the mean direction and the median direction. We also derive some properties of our measure by the use of the geometric nature of a circle.

1. Introduction

There are three basic approaches to directional statistics, (i) embedding, (ii) wrapping and (iii) intrinsic approaches. For a discussion of these approaches, see Jupp and Mardia [1989]. These are usually used in different areas, depending on their own merits. For examples, the embedding approach is mainly used for inferential problems, see Watson [1983]. This is because the embedding approach is comparatively easy to carry out various calculations. But this approach possesses an outer space in a sense that the dimension of the space considered is higher than that of the original space, and hence in some cases the results contradict our intuition. That is why we try to define a natural measure corresponding to the 'mean' intrinsically. For applications of distributions on $\mathscr{S}^1$, see Fisher [1993].

Throughout this paper we identify the unit circle $\mathscr{S}^1 = \{(x, y) \in \mathscr{R}^2 | x^2 + y^2 = 1\}$ with a quotient space $\mathscr{R}/2\pi\mathscr{Z}$, and consider the quotient map

$$q: \mathscr{R} \to \mathscr{S}^1 = \mathscr{R}/2\pi\mathscr{Z} ,$$

where $\mathscr{R}$ and $\mathscr{Z}$ denote the sets of real numbers and integers, respectively. For $\theta \in \mathscr{S}^1$ and $s \in \mathscr{R}$, we define a real number $x_\theta^{(s)}$ as a unique point such that $q(x_\theta^{(s)}) = \theta$, and $x_\theta^{(s)} \in (s - \pi, s + \pi] \subset \mathscr{R}$. Then we can define the function $r: \mathscr{S}^1 \to (0, 2\pi] \subset \mathscr{R}$ as

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