HIROSHIMA MATH. J.
26 (1996), 277–287

On a characterization of L^p-norm and a converse of Minkowski's inequality

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(Received February 18, 1991)

ABSTRACT. Let **C** be a cone in a linear space. Under some weak regularity conditions we show that every subadditive function $\mathbf{p}: \mathbf{C} \to \boldsymbol{R}$ such that $\mathbf{p}(rx) \leqslant r\mathbf{p}(x)$ for some $r \in (0, 1)$ and all $x \in \mathbf{C}$ must be positively homogenous. As an application we obtain a new characterization of L^p-norm. This permits us to prove among other things the following converse of Minkowski's inequality.

Let (Ω, Σ, μ) be a measure space such that there exist disjoint sets $A, B \in \Sigma$ satisfying the condition $\mu(B) = 1/\mu(A)$, $\mu(A) \neq 1$. If $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is an arbitrary bijection such that

$$\varphi^{-1}\left(\int_\Omega \varphi \circ (x + y) d\mu\right) \leqslant \varphi^{-1}\left(\int_\Omega \varphi \circ x d\mu\right) + \varphi^{-1}\left(\int_\Omega \varphi \circ y d\mu\right)$$

for all the μ-integrable step functions $x, y: \Omega \to \boldsymbol{R}_+$ then φ is a power function.

Introduction

Let $\boldsymbol{R}$, $\boldsymbol{R}_+$ and $\boldsymbol{N}$ denote respectively the set of reals, nonnegative reals and positive integers.

For a measure space (Ω, Σ, μ) let $\mathbf{S} = \mathbf{S}(\Omega, \Sigma, \mu)$ stand for the linear space of all the μ-integrable step functions $x: \Omega \to \boldsymbol{R}$ and let $\mathbf{S}_+ := \{x \in S: x \geqslant 0\}$.

It can be easily verified that for every bijection $\varphi: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ such that $\varphi(0) = 0$ the functional $\boldsymbol{P}_\varphi: \mathbf{S} \to \boldsymbol{R}_+$ given by the formula

$$\boldsymbol{P}_\varphi(x) := \varphi^{-1}\left(\int_\Omega \varphi \circ |x| d\mu\right), \qquad x \in \mathbf{S}, \tag{1}$$

is well defined. In [4] we have proved the following *converse of Minkowski's inequality*.

Let (Ω, Σ, μ) be a measure space with two sets $A, B \in \Sigma$ such that

$$0 < \mu(A) < 1 < \mu(B) < \infty \tag{2}$$

1991 *Mathematics Subject Classification.* 46E30, 26D15, 39B72.

Key words and phrases. Subadditive functions on a cone, homogeneity, measure space, characterization of L^p-norm, power function.