HIROSHIMA MATH. J.
27 (1997), 177–188

Higher Specht polynomials

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(Received September 14, 1995)
(Revised February 23, 1996)

ABSTRACT. A basis of the quotient ring S/J_+ is given, where S is the ring of polynomials and J_+ is the ideal generated by symmetric polynomials of positive degree. They are called higher Specht polynomials.

0. Introduction

The purpose of this paper is to give a detailed proof of the result announced in [4], and to give its generalization.

Let $S = \mathbf{C}[x_0, \dots, x_{n-1}]$ be the algebra of polynomials of n variables $x_0, \dots, x_{n-1}$ with complex coefficients, on which the symmetric group $\mathfrak{S}_n$ acts by the permutation of the variables:

$$(\sigma f)(x_0, \dots, x_{n-1}) = f(x_{\sigma(0)}, \dots, x_{\sigma(n-1)})(\sigma \in \mathfrak{S}_n)$$

Let $e_j(x_0, \dots, x_{n-1}) = \sum_{0 \le i_1 < \cdots < i_j \le n-1} x_{i_1} \dots x_{i_j}$ be the elementary symmetric polynomial of degree j and set $J_+ = (e_1, \dots, e_n)$, the ideal generated by $e_1, \dots, e_n$. The quotient ring $R = S/J_+$ has a structure of an $\mathfrak{S}_n$-module. Let $n_0, \dots, n_{r-1}$ be natural numbers such that $n = \sum_{i=0}^{r-1} n_i$. Then the product of symmetric groups $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$ is naturally embedded in $\mathfrak{S}_n$. By restricting to this subgroup, R is an $\mathfrak{S}_{n_0} \times \cdots \times \mathfrak{S}_{n_{r-1}}$-module. We give a combinatorial procedure to obtain a basis of each irreducible component of R. In view of this construction, these polynomials such obtained might be called higher Specht polynomials. The case $n_0 = n$ is treated in [4]. When $n_0 = \cdots = n_{n-1} = 1$, this basis becomes the descent basis for R (see [3]).

As an application, we also give a similar basis for a complex reflection group $G_{r,n} = (\mathbf{Z}/r\mathbf{Z}) \wr \mathfrak{S}_n$. Let S be the symmetric algebra of the natural $G_{r,n}$ representation over $\mathbf{C}$. The ring of invariants $S^{G_{r,n}}$ is known to be isomorphic to a polynomial ring $\mathbf{C}[e_1^{(r)}, \dots, e_n^{(r)}]$ generated by the elementary symmetric polynomials $e_1^{(r)}, \dots, e_n^{(r)}$ in x_i^r's. We put $R^{(r)} = S/J_+$, where $J_+ = (e_1^{(r)}, \dots, e_n^{(r)})$. As a $G_{r,n}$-module, it is equivalent to the regular representation. It is also known that the irreducible representations of $G_{r,n}$ are indexed

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1991 *Mathematics Subject Classification.* 05E10, 20C30.
Key words and phrases. Reflection groups, Specht polynomials.