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First order partial differential equations on the curvature of 3-dimensional Heisenberg bundles

Dedicated to Professor Yoshihiro Tashiro on his 70th birthday

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ABSTRACT. We study first order partial differential equations on the curvature of principal fibre bundles. We show that such differential equations are essentially exhausted by the one obtained from the Bianchi identity, and as one example, we express the differential equations in the case of 3-dimensional Heisenberg bundles in a geometric form. In the latter half of this paper, we study some algebraic properties concerning the Bianchi identity for 3-dimensional Heisenberg bundles. Several types of invariants and covariants naturally arise from studying this algebraic problem.

Introduction

"Prescribed curvature problem", i.e., the problem of characterizing "actual" curvature tensor fields (or forms) among the set of curvature like tensor fields (or forms), is one of the fundamental problem in differential geometry, and also in physics. In general, not all curvature like tensor fields are actually curvature, and several results are known at present concerning this problem for each geometric situation. For example, in a series of papers, Kazdan and Warner characterized the curvature functions on 2-dimensional manifolds from global viewpoints [8], [9], while local characterizations of curvatures are studied deeply in [3], [5], [6], [7], [13], etc.

If Ω is an actual curvature determined by a connection, the components of Ω must satisfy some partial differential equations. As a classically known example, in the context of principal G-bundles, the characteristic form $f(\Omega)$ corresponding to a G-invariant polynomial f is closed, and we may consider the equality $df(\Omega) = 0$ as a first order partial differential equation on Ω. It is also known that in the case of $SU(2)$-bundle over $\boldsymbol{R}^4$, the curvature like form Ω which satisfies some second order partial differential equations is an

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