HIROSHIMA MATH. J.
30 (2000), 205–213

A form of classical Liouville theorem for polyharmonic functions

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(Received May 6, 1999)
(Revised August 2, 1999)

ABSTRACT. Our aim in this paper is to propose a form of classical Liouville theorem for polyharmonic functions which is a direct generalization of our former one for harmonic functions.

1. Introduction

We denote by $\mathbf{R}^d$ the Euclidean space of dimension $d \geq 2$. The length $|x|$ of a point or a vector $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$ is given by $|x| = (\sum_{i=1}^d x_i^2)^{1/2}$. A real valued function $u(x)$ is *harmonic* on $\mathbf{R}^d$ if $u \in C^2(\mathbf{R}^d)$ and $\Delta u(x) = 0$ on $\mathbf{R}^d$, where Δ is the Laplacian $\sum_{i=1}^d (\partial/\partial x_i)^2$. We denote by $H(\mathbf{R}^d)$ the real linear space of harmonic functions on $\mathbf{R}^d$. We also denote by $HB(\mathbf{R}^d)$ ($HP(\mathbf{R}^d)$, resp.) the class of bounded (nonnegative, resp.) functions $u \in H(\mathbf{R}^d)$. The Liouville theorem in the theory of harmonic functions (cf. e.g. Axler et al. [3]) consists of the following two contents: $HB(\mathbf{R}^d) = \mathbf{R}$; $HP(\mathbf{R}^d) = \mathbf{R}^+$, where $\mathbf{R}$ is the real number field and $\mathbf{R}^+ = \{t \in \mathbf{R} : t \geq 0\}$. Picard (cf. e.g. [11], [12]) essentially showed that these two statements are equivalent. We proposed ([8]) the following theorem (stated here in a slightly modified fashion beyond the original presentation) as a form of Liouville theorem for harmonic functions.

THEOREM A. *Suppose $u \in H(\mathbf{R}^d)$ and s is any real number with $s > 0$. Then u is a harmonic polynomial of degree less than s if and only if there exists an increasing divergent sequence $(r_i)_{i \geq 1}$ of positive numbers r_i $(i = 1, 2, \ldots)$ such that*

$$\liminf_{i \uparrow \infty} \left(\min_{|x| = r_i} \frac{u(x)}{|x|^s} \right) \geq 0. \tag{1.1}$$

In fact, if $u \in HB(\mathbf{R}^d)$ ($u \in HP(\mathbf{R}^d)$, resp.), then (1.1) is valid for any $0 < s < 1$ and for any increasing divergent positive sequence $(r_i)_{i \geq 1}$. Thus

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2000 *Mathematics Subject Classification.* Primary 31B30.

Key words and phrases. Harmonic, polyharmonic, Liouville theorem, spherical harmonics, Fourier expansion, Almansi decomposition, polynomial.

This work is partly supported by Grant-in-Aid for Scientific Research, Nos. 09640230, 10640190, Japanese Ministry of Education, Science and Culture.