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Homotopy spherical space forms—a numerical bound for homotopy types

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ABSTRACT. Let G be a finite group. We show that for a fixed $n \geq 1$ the set of homotopy types of orbit spaces of all free G-actions on homotopy $(2n-1)$-spheres is finite and bounded by the order of some quotient group associated with G. In particular, we deduce that there are at most two homotopy types of lens spaces determined by all free $\mathbf{Z}/p^m$-actions on homotopy 3-spheres when p is an odd prime, and only one homotopy type of those spaces provided that $4 \nmid p-1$. There is also only one homotopy type of lens spaces of dimension $2n-1$ determined by all $\mathbf{Z}/2^m$-free actions provided that n is odd.

Introduction

A finite group has the periodic homology if it acts freely on a finite homology sphere [1, Chap. III] and [3, 4]. On the other hand, Swan [10] showed that any finite group with periodic homology of period l acts freely on a finite CW complex of the homotopy type of an $(nl-1)$-sphere for some positive integer n. This CW complex may not have the homotopy type of a closed manifold. The symmetric group S_3 has a periodic homology but does not act freely on any manifold which has the homotopy type of a sphere due to the result of Milnor [8]: *Every element of order two must be in the center*.

The study of free actions of a group on homotopy spheres is related to the study of their orbit spaces. In the case of the cyclic group $\mathbf{Z}/2$ of order two, the orbit space has the homotopy type of the real projective space and the problem of classifying manifolds of this homotopy type has been studied extensively in [7, 11]. An old result of homotopy type theory (see e.g. [9]) says that, up to homotopy, the set of lens spaces exhausts all the homotopy types of orbit spaces of free actions of the cyclic group $\mathbf{Z}/m$ of order m on homotopy $(2n-1)$-spheres. Thus the classification of all the $\mathbf{Z}/m$-actions on those homotopy spheres is equivalent to the classification of manifolds of the homotopy type of lens spaces studied by Browder in [2]. The set of homotopy

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