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A Liouville theorem for polyharmonic functions

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ABSTRACT. We give a short, elementary proof of a theorem which shows that if u is a polyharmonic function on $\mathbf{R}^d$ and the growth of u^+ is suitably restricted, then u must be a polynomial.

1. Introduction

A typical point of $\mathbf{R}^d$, where $d \geq 2$, is denoted by $x = (x_1, \dots, x_d)$. We write Δ for the Laplace operator $\sum_{j=1}^{d} \partial^2/\partial x_j^2$ and define $\Delta^1 = \Delta$ and $\Delta^{p+1} = \Delta\Delta^p$ when $p \in \mathbf{N}$. A function $u : \mathbf{R}^d \to \mathbf{R}$ is called *polyharmonic* of order p if $u \in C^{2p}(\mathbf{R}^d)$ and $\Delta^p u \equiv 0$ on $\mathbf{R}^d$. We denote the vector space of all such functions by $\mathscr{H}^p$. Thus, in particular, $\mathscr{H}^1$ is the space of all harmonic functions on $\mathbf{R}^d$. The positive part of a function $u : \mathbf{R}^d \to \mathbf{R}$ is denoted by u^+; that is, $u^+(x) = \max\{u(x), 0\}$ for each x in $\mathbf{R}^d$.

A classical Liouville theorem for harmonic functions may be stated as follows: if $u \in \mathscr{H}^1$ and u^+ is bounded on $\mathbf{R}^d$, then u is constant. A generalization, due to Kuran [4, Theorem 2], shows that if $u \in \mathscr{H}^p$ and u^+ is bounded on $\mathbf{R}^d$, then u is a polynomial of degree at most $2p - 2$. Several authors have given further generalizations. To describe their results, we introduce some more notation. The open ball and the sphere of radius r centred at the origin 0 of $\mathbf{R}^d$ are denoted by $B(r)$ and $S(r)$. We denote d-dimensional Lebesgue measure by λ and $(d-1)$-dimensional surface measure by σ. Some known results are summarized in the following theorem.

THEOREM A. *Let $u \in \mathscr{H}^p$, where $p \in \mathbf{N}$, and let s be a number such that $s > 2p - 2$. The following statements are equivalent:*

(1) *u is a polynomial of degree less than s;*

(2) $\lim_{r\to+\infty} r^{-s-d+1} \int_{S(r)} u^+ \, d\sigma = 0$;

(3) $\liminf_{r\to+\infty} r^{-s-d} \int_{B(r)} u^+ \, d\lambda = 0$;

(4) $\liminf_{r\to+\infty} (r^{-s} \max\{u^+(x) : x \in S(r)\}) = 0$.

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