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Congruent numbers over real quadratic fields

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ABSTRACT. Let m $(\neq 1)$ be a square-free positive integer. We say that a positive integer n is a congruent number over $\mathbf{Q}(\sqrt{m})$ if it is the area of a right triangle with three sides in $\mathbf{Q}(\sqrt{m})$. We put $K = \mathbf{Q}(\sqrt{m})$. We prove that if $m \neq 2$, then n is a congruent number over K if and only if $E_n(K)$ has a positive rank, where $E_n(K)$ denotes the group of K-rational points on the elliptic curve E_n defined by $y^2 = x^3 - n^2 x$. Moreover, we classify right triangles with area n and three sides in K.

1. Introduction

A positive integer n is called a congruent number if it is the area of a right triangle whose three sides have rational lengths. For each positive integer n, let E_n be the elliptic curve over $\mathbf{Q}$ defined by $y^2 = x^3 - n^2 x$, and $E_n(k)$ the group of k-rational points on E_n for a number field k. By the following well-known theorem, we have a condition such that n is a congruent number in terms of $E_n(\mathbf{Q})$.

THEOREM A (cf. [**4**, p. 46]). *A positive integer n is a congruent number if and only if $E_n(\mathbf{Q})$ has a point of infinite order.*

Let ∞ be the point at infinity of $E_n(\mathbf{Q})$ which is regarded as the identity for the group structure on E_n. We note that, in the proof of Theorem A, we use that the torsion subgroup of $E_n(\mathbf{Q})$ consists of four elements ∞, $(0,0)$, and $(\pm n, 0)$ of order 1 or 2.

For any positive integer n, determining whether it is a congruent number or not is a classical problem. In relation to Theorem A, some important results are known. By the result of J. Coates and A. Wiles [**2**] for elliptic curves E over $\mathbf{Q}$ with complex multiplication, if the rank of $E_n(\mathbf{Q})$ is positive, then $L(E_n, 1) = 0$, where $L(E_n, s)$ is the Hasse-Weil L-function of $E_n/\mathbf{Q}$. Assuming the weak Birch and Swinnerton-Dyer conjecture [**1**], it is known that if $L(E_n, 1) = 0$, then the rank of $E_n(\mathbf{Q})$ is positive. F. R. Nemenzo [**7**] showed that for $n < 42553$, the weak Birch and Swinnerton-Dyer conjecture holds for E_n, i.e.,

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