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# Cobordism group with local coefficients and its application to 4-manifolds

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**ABSTRACT.** For a pair $(X, A)$ of topological spaces and $w \in H^1(X; \mathbf{Z}_2)$ the cobordism group $\Omega_n(X, A; \mathscr{S}_w)$ with local coefficients is introduced. If $X$ is a CW complex and $\mathscr{S}_w$ is a local system over $X$ determined by $w$, then we have an Atiyah-Hirzeburch spectral sequence $E^2_{p,q} = H_p(X; \Omega_q \otimes \mathscr{S}_w) \Rightarrow \Omega_{p+q}(X; \mathscr{S}_w)$ which is regular and hence convergent. For a connected CW complex $X$ the map $\mu : \Omega_4(X; \mathscr{S}_w) \to H_4(X; \mathscr{S}_w)$, defined by $\mu([M, f, \varphi]) = f_*(\varphi_*(\sigma))$, is a surjection and its kernel is $\Omega_4 \otimes \mathbf{Z}_2$ if $w \neq 0$, where $\sigma$ is a fundamental homology class with respect to the orientation sheaf of a manifold $M$ and $\varphi$ is a local orientation. The closed 4-manifolds with finitely presentable fundamental group $\pi$ and the first Stiefel-Whitney class induced from $w$ are almost classified modulo connected sums with simply connected manifolds by the quotient $H_4(B\pi; \mathscr{S}_w)/(\mathrm{Aut}\,\pi)^w_*$, and precisely in the case that $\pi$ is abelian.

## 1. Introduction

The oriented cobordism functor $\{\Omega_*(X, A), \varphi_*, \partial\}$ satisfies the first six Eilenberg-Steenrod axioms for the category of pairs of topological spaces and maps [2]. So, for any CW complex $X$ the Atiyah-Hirzeburch spectral sequence

$$E^2_{p,q} = H_p(X; \Omega_q) \Rightarrow \Omega_{p+q}(X)$$

is regular and hence convergent in the sense of [1]. Using this spectral sequence, the classification of oriented closed 4-manifolds having the finitely presentable fundamental group $\pi$ modulo connected sums with simply connected manifolds is given by the quotient $H_4(B\pi; \mathbf{Z})/(\mathrm{Aut}\,\pi)_*$ [4], [7].

Our goal of this paper is to extend the above result to the non-orientable case. We introduce a cobordism group $\Omega_n(X, A; \mathscr{S}_w)$ for a pair $(X, A)$ of topological spaces and $w \in H^1(X; \mathbf{Z}_2)$, which reduces to $\Omega_n(X, A)$ if $w = 0$. Let $w_1 : BO_r \to K(\mathbf{Z}_2, 1)$ be the map corresponding to the first Stiefel-Whitney class. Consider $w$ to be a map of $X$ to $K(\mathbf{Z}_2, 1)$, and let

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