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The structure of the Hecke algebras of $GL_2(F_q)$ relative to the split torus and its normalizer

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Abstract. Let A be the subgroup of $G = GL_2(F_q)$ consisting of diagonal matrices. We study the structure of the Hecke algebra $\mathscr{H}(G, A)$ of G relative to A. In particular, we determine the multiplication table of $\mathscr{H}(G, A)$ with respect to the standard basis. As an application, we describe the multiplication table of the Hecke algebra $\mathscr{H}(G, H)$ where H is the normalizer of A in G.

1. Introduction

The Hecke algebra $\mathscr{H}(G, A)$ of a finite group G relative to its subgroup A is a generalization of the group algebra $\mathbf{C}G$ of G, whose structure and representations are interesting mathematical objects as well as those of $\mathbf{C}G$.

In particular, the Hecke algebra $\mathscr{H}(G, A)$ plays an important role in the study of vertex-transitive graphs with vertex set G/A. In fact, such a graph is constructed by giving a certain family of double cosets of G relative to A. Moreover the adjacency matrix and its powers of such a graph are described in terms of the elements of $\mathscr{H}(G, A)$ ([3]). Therefore if one knows the multiplicative structure and irreducible characters of $\mathscr{H}(G, A)$, one can find the spectra of vertex-transitive graphs over G/A.

Let $G = GL_2(F_q)$ be the general linear group of 2×2 non-singular matrices over the finite field F_q, and let A be the subgroup of diagonal matrices of G (a split torus of G) and H be the normalizer of A in G. In our previous paper ([4]), we have considered the irreducible characters of $\mathscr{H}(G, A)$ and described the character table of it with respect to the standard basis of $\mathscr{H}(G, A)$. In the present article, we study the multiplicative structure of both $\mathscr{H}(G, A)$ and $\mathscr{H}(G, H)$. In particular we determine the multiplication tables of both $\mathscr{H}(G, A)$ and $\mathscr{H}(G, H)$ with respect to their standard basis.

The paper is organized as follows. In §2 we consider the double coset spaces $A\backslash G/A$ and $H\backslash G/H$. Using Bruhat decomposition of G, we determine a complete set $\mathscr{R}$ of representatives of $A\backslash G/A$ in Theorem 2.1. Moreover decomposing an H double coset into A double cosets, we give a complete set of

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