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On products of distributions involving delta function and its partial derivatives

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Abstract. Li Banghe and Li Yaqing defined in [1] the product $S \circ T$ as a hyperdistribution (we call hyperdistributions to the complex linear functional of $\mathscr{D}(\boldsymbol{R}^n)$ in ${}^\rho\boldsymbol{C}'$) for S and T in $D'(\boldsymbol{R}^n)$ and they calculated $\delta \circ \delta$. In this paper, we shall obtain expressions for the products $\delta \circ \frac{\partial}{\partial x_i}\delta$ and $\frac{\partial}{\partial x_i}\delta \circ \frac{\partial}{\partial x_i}\delta$ for $i = 1, \dots, n$, and for even n, these products have the Hadamard finite part nonzero.

1. Introduction

In this section we shall define the product "$\circ$". This definition involves a harmonic representation of distributions and Non-Standard Analysis. Let $\boldsymbol{C}^*$ and $\boldsymbol{R}^*$ be nonstandard models for the complex field and the real field respectively, and let ρ be a positive infinitesimal. Let Θ denote the set of all infinitesimal in $\boldsymbol{C}^*$. Let

$$ {}^\rho\boldsymbol{C} = \{x \in \boldsymbol{C}^* \colon \text{for some finite integer } n, |x| < \rho^{-n}\}, \quad {}^\rho\boldsymbol{C}' = {}^\rho\boldsymbol{C}/\Theta. $$

We call *hyperdistributions* the complex linear functional of $\mathscr{D}(\boldsymbol{R}^n)$ in ${}^\rho\boldsymbol{C}'$.

Definition 1.1. *Let $T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $u(x, y)$ be a harmonic function in $\boldsymbol{R}^{n+1}_+ = \{(x, y) : x \in \boldsymbol{R}^n, y > 0\}$ so that $\lim_{y\to 0_+} u(x, y) = T$ in the sense of $\mathscr{D}'(\boldsymbol{R}^n)$. Then $u(x, y)$ is called a harmonic function associated with T or* harmonic representation *of T.*

Lemma 1.2. *Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$ and $\hat{S}, \hat{T}$ be harmonic representations of S and T respectively, and ${}^*\hat{S}, {}^*\hat{T}$ denote the nonstandard extensions of $\hat{S}, \hat{T}$ respectively. For any $\phi \in \mathscr{D}(\boldsymbol{R}^n)$,*

$$ \langle {}^*\hat{S}(x, \rho)^*\hat{T}(x, \rho), {}^*\phi(x) \rangle \in {}^\rho\boldsymbol{C}. $$

Lemma 1.3. *Let $S, T \in \mathscr{D}'(\boldsymbol{R}^n)$. If $\hat{S}_1, \hat{S}_2$ and $\hat{T}_1, \hat{T}_2$ are two harmonic representations of S and T respectively, then for any $\phi \in \mathscr{D}(\boldsymbol{R}^n)$, we have*

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