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Remarks on universal functions of $\mathcal{O}(\mathbf{C}^*)$

Makoto Abe, Teisuke Jin and Tadashi Shima

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Abstract. Let $A(\mathbf{C}^*)$ be the family of all $\mathcal{O}(\mathbf{C}^*)$-convex compact sets of $\mathbf{C}^*$ and $B(\mathbf{C}^*)$ the family of all compact sets of $\mathbf{C}^*$ whose complements in $\mathbf{C}^*$ are connected. Then the family $B(\mathbf{C}^*)$ is the maximal subfamily of $A(\mathbf{C}^*)$ on which there exists a universal function of $\mathcal{O}(\mathbf{C}^*)$. We also prove the transcendence of the universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $B(\mathbf{C}^*)$.

1. Introduction and preliminaries

Let X be a complex manifold. We denote by $\mathcal{O}(X)$ the set of all holomorphic functions on X. For any compact set K of X the set

$$\hat{K}_X := \left\{ z \in X \mid |f(z)| \le \max_{x \in K} |f(x)| \text{ for every } f \in \mathcal{O}(X) \right\}$$

is said to be the *holomorphically convex hull* of K in X. A compact set K of X is said to be $\mathcal{O}(X)$*-convex* if $\hat{K}_X = K$. According to Zappa [8] we denote by $A(X)$ the family of all $\mathcal{O}(X)$-convex compact sets of X.

Let G be a Stein group (see for example Grauert-Remmert [5, p. 136]) and $\mathscr{S}$ a subfamily of $A(G)$. A function $F \in \mathcal{O}(G)$ is said to be a *universal function* of $\mathcal{O}(G)$ on $\mathscr{S}$ if for every $f \in \mathcal{O}(G)$, $K \in \mathscr{S}$ and $\varepsilon > 0$ there exists an element $c \in G$ such that $\max_{x \in K}|F(c \cdot x) - f(x)| < \varepsilon$.

For the additive group $\mathbf{C}^n$, $n \ge 1$, there exists a universal function of $\mathcal{O}(\mathbf{C}^n)$ on $A(\mathbf{C}^n)$ by Birkhoff [4], Luh [6], Y. Abe [1] and Abe-Zappa [3]. For the multiplicative group $\mathbf{C}^* = GL(1, \mathbf{C}) = \mathbf{C} - \{0\}$ there exist no universal functions of $\mathcal{O}(\mathbf{C}^*)$ on $A(\mathbf{C}^*)$ by Remark 2 of Zappa [8, p. 350]. For the complex general linear group $GL(n, \mathbf{C})$, $n \ge 2$, it is not known whether there does exist a universal function of $\mathcal{O}(GL(n, \mathbf{C}))$ on $A(GL(n, \mathbf{C}))$ or not (see Abe-Zappa [3, p. 231]).

According to Zappa [8] let $B(\mathbf{C}^*)$ be the family of all compact sets K of $\mathbf{C}^*$ such that $\mathbf{C}^* - K$ is connected. Here we remark that $B(\mathbf{C}^*)$ is a proper

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