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An umbilical point on a non-real-analytic surface

Naoya ANDO

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ABSTRACT. Let F be a smooth function of two variables which is zero at $(0,0)$ and positive on a punctured neighborhood of $(0,0)$. Then the function $\exp(-1/F)$ is smoothly extended to $(0,0)$ and then the origin o of $\mathbf{R}^3$ is an umbilical point of its graph. In this paper, we shall study the behavior of the principal distributions around o on condition that the norm of the gradient vector field of $\log F$ is bounded from below by a positive constant on a punctured neighborhood of $(0,0)$.

1. Introduction

Let S be a surface in $\mathbf{R}^3$ and p_0 an isolated umbilical point of S. Then the *index* of p_0 on S is defined by the index of p_0 with respect to a principal distribution.

It is known that if S is a surface with constant mean curvature and if S is connected and not totally umbilical, then each umbilical point of S is isolated and its index is negative ([Ho, p139]); if S is a special Weingarten surface, then the same result is obtained ([HaW]).

It has been expected that the index of an isolated umbilical point on a surface is not more than one. We call this conjecture the *index conjecture*. In relation to the index conjecture, the following two conjectures are known: Carathéodory's conjecture and Loewner's conjecture. *Carathéodory's conjecture* asserts that there exist at least two umbilical points on a compact, strictly convex surface in $\mathbf{R}^3$. If the index conjecture is true, then we see from Hopf-Poincaré's theorem that there exist at least two umbilical points on a compact, orientable surface of genus zero, and this immediately gives the affirmative answer to Carathéodory's conjecture. Let F be a real-valued, smooth function of two real variables x, y, and set $\partial_{\bar{z}} := (\partial/\partial x + \sqrt{-1}\partial/\partial y)/2$. Then *Loewner's conjecture* for a positive integer $n \in \mathbf{N}$ asserts that if a vector field

$$\mathrm{Re}(\partial_{\bar{z}}^n F)\frac{\partial}{\partial x} + \mathrm{Im}(\partial_{\bar{z}}^n F)\frac{\partial}{\partial y}$$

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