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Retractions of H-spaces

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ABSTRACT. Stasheff showed that if a map between H-spaces is an H-map, then the suspension of the map is extendable to a map between projective planes of the H-spaces. Stahseff also proved the converse under the assumption that the multiplication of the target space of the map is homotopy associative. We show by giving an example that the assumption of homotopy associativity of the multiplication of the target space is necessary to show the converse. We also show an analogous fact for maps between A_n-spaces.

1. Introduction

Let X and Y be H-spaces, and $f : X \to Y$ a map. Stasheff [4] showed that if f is an H-map, then it's suspension $\Sigma f : \Sigma X \to \Sigma Y$ is extendable to a map $P_2 f : P_2 X \to P_2 Y$ between projective planes $P_2 X$ and $P_2 Y$ of X and Y, respectively. He also showed the converse under the assumption that the multiplication μ_Y of Y is homotopy associative. It has not been known whether the converse holds without the assumption of the homotopy associativity of μ_Y. In this paper we show by giving an example that the assumption of homotopy associativity of μ_Y is necessary to show the converse.

Our example is the retraction $r : J(X) \to X$ for an H-space X. Here, $J(X)$ is the reduced power space of X introduced by James [2], which has the homotopy type of $\Omega\Sigma X$. By definition $J(X)$ is an identification space of $\bigcup_{i \geq 1} X^i$. Then the map r is defined by

$$r([x_1, \dots, x_i]) = (\cdots((x_1 \cdot x_2) \cdot x_3) \cdots) \cdot x_i,$$

where $[x_1, \dots, x_i]$ is the class of $(x_1, \dots, x_i) \in X^i$ and $x \cdot y$ denotes the multiplication of x and y. Our result is stated as follows.

THEOREM 1.1. *For any H-space X, there is an extension $P_2 r : P_2 J(X) \to P_2 X$ of $\Sigma r : \Sigma J(X) \to \Sigma X$.*

Stasheff showed the following

THEOREM 1.2 ([4]). *The retraction r is an H-map if and only if the multiplication of X is homotopy associative.*