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ON THE ZERO-ONE SET OF AN ENTIRE FUNCTION

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1. Introduction. Let $\{a_n\}$ and $\{b_n\}$ be two disjoint infinite sequences with no finite limit points. If it is possible to construct an entire function f whose zero sequence is exactly $\{a_n\}$ and whose one sequence is exactly $\{b_n\}$, the pair $(\{a_n\}, \{b_n\})$ is called the zero-one set of f. In general an arbitrary pair of two sequences $\{a_n\}$, $\{b_n\}$ is not a zero-one set of any entire function. This was recently proved by Rubel and Yang [5] explicitly. On giving $\{a_n\}$ they constructed $\{b_n\}$ in a very skillful but artificial manner. It seems to the present author that their $\{b_n\}$ has less arbitrariness in a sense and has too much arbitrariness in the other sense. In this paper we shall discuss the following problem: How can it be arbitrary? Our answer is given in the following.

THEOREM 1. *Let $\{a_n\}$ and $\{b_n\}$ be two arbitrary disjoint infinite sequences with no finite limit points. Let b_1 be different from b_2. Then one of the following three pairs*

$$(\{a_n\}, \{b_n\}_{n=1}^{\infty}), (\{a_n\}, \{b_n\}_{n=2}^{\infty}), (\{a_n\}, \{b_n\}_{n=3}^{\infty} \cup \{b_1\})$$

is not a zero-one set of any entire function.

THEOREM 2. *Suppose that $(\{a_n\}, \{b_n\})$ is the zero-one set of an entire function $N(z)$ of finite non-integral order. Then $(\{a_n\}, \{b_n\}_{n=2}^{\infty})$ is not a zero-one set of any entire function.*

We shall give an example showing that two pairs are really zero-one sets in Theorem 1 and that the finite nonintegrity assumption cannot be omitted in Theorem 2. We shall give other several examples being connected with closely related problems. Our method of proof depends also upon the impossibility of the Borel identity, which had been stated in several ways. See [1], [2], [3], [4]. Theorem 1 corresponds to the so-called three function theorem.

2. Proof of Theorem 1. Suppose that all of the given pairs are zero-one sets. Then there are entire functions N, f and g satisfying

1) $f=Ne^{\alpha}$, $(f-1)(z-b_1)=(N-1)e^{\beta}$,

$g=Ne^{\gamma}$, $(g-1)(z-b_2)=(N-1)e^{\delta}$,

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