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ON ORISPHERICAL SUBGROUPS OF A SEMISIMPLE LIE GROUP

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1. Let $\mathfrak{g}$ be a real semisimple Lie algebra, and let $\mathcal{G}$ be a Lie group whose Lie algebra is $\mathfrak{g}$. We take a one-parameter subgroup $g(t)=\exp(tX)$, $X\in\mathfrak{g}$, and define a orispherical subgroup $\mathcal{Z}$ relative to $g(t)$ as follows.

DEFINITION 1. $\mathcal{Z}$ is the set of all $z\in\mathcal{G}$ for which

$$\lim_{t\to\infty} g(t)zg(t)^{-1}=e \quad \text{(neutral element of } \mathcal{G}).$$

Orispherical subgroups were introduced by Gelfand, Graev, and Pyatetskii-Shapiro, and played an important role in the theory of representations and automorphic functions; [1], [2]. The purpose of this note is to show that $\mathcal{Z}$ is a connected closed subgroup of $\mathcal{G}$.

2. First of all, $\mathcal{Z}$ is easily seen to be connected. Let $z\in\mathcal{Z}$, then $\exp(tX)\cdot z\cdot\exp(-tX)=z_t\in\mathcal{Z}$ by definition. Of course z_t is continuous in t, and $z_t\to e$ as $t\to\infty$. Denote by $\mathcal{Z}_0$ the connected component of e of $\mathcal{Z}$. Since $\mathcal{Z}_0$ is open in $\mathcal{Z}$, we have $z_{t_0}\in\mathcal{Z}_0$ for sufficiently large t_0. But then z_t $(0\leqq t\leqq t_0)$ connects z to z_{t_0}, hence $z\in\mathcal{Z}_0$. This proves connectedness of $\mathcal{Z}$.

3. It is a classical result that any $X\in\mathfrak{g}$ can be expressed by a unique sum $Y+N$, where $Y, N\in\mathfrak{g}$ satisfy the conditions: i) $[Y, N]=0$; ii) ad Y is semisimple and all of its eigen values are real; iii) ad N has only pure imaginary eigen values. Here ad means the adjoint representation of $\mathfrak{g}$.

Let Ad be the adjoint representation of $\mathcal{G}$ into the set of Aut $(\mathfrak{g})$ of all automorphisms of the Lie algebra $\mathfrak{g}$. We denote the image Ad $\mathcal{G}$ of $\mathcal{G}$ by Int $(\mathfrak{g})$ Then it is obvious that if $z\in\mathcal{Z}$, we have

$$\lim_{t\to\infty} \mathrm{Ad}(\exp(tX))\cdot\mathrm{Ad}\,z\cdot\mathrm{Ad}(\exp(-tX))=E \quad \text{(Identity).}$$

In Int $(\mathfrak{g})$, we consider the subset $\boldsymbol{Z}$ of all $\zeta\in\mathrm{Int}(\mathfrak{g})$ for which

$$\lim_{t\to\infty} \mathrm{Ad}(\exp(tX))\cdot\zeta\cdot\mathrm{Ad}(\exp(-tX))=E\,.$$

Now we put ad $X=\xi$, ad $Y=\eta$, ad $N=\nu$, then

$$\xi=\eta+\nu, \qquad \eta\nu=\nu\eta,$$
$$\mathrm{Ad}(\exp(tX))=e^{t\xi}=e^{t\nu}e^{t\eta}$$

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