MEAN CONVERGENCE OF A FOURIER SERIES AND A FOURIER TRANSFORM

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1. Let $f(x)$ be a function of $L_1(-\pi,\pi)$, and suppose that $f(x)$ belongs to L_r on a subinterval (a,b) of $(-\pi,\pi)$, $r>1$. We consider about the convergence in mean (L_r) over (a,b) of the Fourier series of $f(x)$. It is well known that if (a,b) coincides with $(-\pi,\pi)$, then the Fourier series of $f(x)$ converges to $f(x)$ in mean L_r over $(-\pi,\pi)$.

Thus if we define $\lambda(x)$ as unity in (a,b) and o outside of (a,b), then the Fourier series of $f(x)\lambda(x)$ which is of $L_r(-\pi,\pi)$ converges in mean (L_r) to $f(x)\lambda(x)$ over $(-\pi,\pi)$, and consequently converges in mean (L_r) to $f(x)$ on (a,b). Since $f(x)\lambda(x)$ and $f(x)$ is identical in (a,b), the Fourier series of $f(x)\lambda(x)$ and of $f(x)$ are uniformly equiconvergent in $(a+\varepsilon, b-\varepsilon)$, ε being a positive number arbitrarily small but fixed. Hence the Fourier series of $f(x)$ converges in mean (L_r) to $f(x)$ in $(a+\varepsilon, b-\varepsilon)$, $\varepsilon>o$.

But the Fourier series of $f(x)$ does not necessarily converge in mean (L_r) on (a,b), which is implied in the fact that will be stated later. (3. (3.5)) And thus we shall consider additional conditions on the behaviors of $f(x)$ at vicinities of $x=a$ and $x=b$ for the mean convergence (L_r) in (a,b).

Also we consider the similar problem in the theory of Fourier transforms. For the sake of convenience we first treat the Fourier transform case. Though we can treat the Fourier series case by similar arguments, we shall deduce it from theorems for Fourier transforms.

2. Theorem 1. *Let $1\leq p\leq 2$ and $p\leq r$, and $f(x)\in L_p(-\infty,\infty)$, $f(x)\in L_r(a,b)$. If there exist constants s_1 and s_2 such that*

$$(2.1)\qquad \int_0^t |f(b+x)-s_2|\,dx = O(t^\alpha),\quad t>o,$$

$$(2.2)\qquad \int_0^t |f(a-x)-s_1|\,dx = O(t^\alpha),\quad t>o,$$

for small t, where $\alpha>1-\frac{1}{r}$, then it holds that

$$(2.3)\qquad \lim_{N\to\infty}\int_a^b |R_N(x)|^r\,dx = o.$$

Here we denote

$$(2.4)\qquad R_N(x) = \int_{-N}^{N}\varphi(t)e^{-ixt}dt - \int_{-N}^{N}\psi(t)e^{-ixt}dt,$$

$\varphi(t)$ and $\psi(t)$ being Fourier transforms of $f(x)$ and $\lambda(x)f(x)$ respectively,

$$(2.5)\qquad \varphi(t)\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixt}f(x)\,dx$$

$$\psi(t)\sim\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ixt}f(x)\lambda(x)\,dx$$

and $\lambda(x)=1$ in $a<x<b$; o outside of (a,b).

From Theorem 2 the following theorem is obtained immediately.

Theorem 2. *Let $1\leq p\leq r\leq 2$. If $f(x)\in L_p(-\infty,\infty)$ and $f(x)\in L_r(a,b)$ and further (2.1) and (2.2) holds for $\alpha>1-\frac{1}{r}$, then*

$$\lim_{N\to\infty}\int_a^b\left|\frac{1}{\sqrt{2\pi}}\int_{-N}^{N}\varphi(t)e^{-ixt}dt - f(x)\right|^r dx = o$$

as $N\to\infty$, where $\varphi(t)$ is the Fourier transform of $f(x)$.

Since $\lambda(x)f(x)\in L_r(-\infty,\infty)\ (r\leq 2)$