ON THE NULL-SET OF A SOLUTION FOR THE EQUATION $\Delta u + k^2 u = 0$

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In the present paper we shall give a certain character of a null-set for the solution of the equation $\Delta u + k^2 u = 0$.

Before we formulate our main proposition we will show the following proposition I which leads us immediately to our main proposition II.

Proposition I. Let m be a set of logarithmic mass zero which is contained in a domain D_0 with boundary C. If a function u, which is continuously differentiable twice and bounded in $D_0 - m$, satisfies an equation $\Delta u + k^2 u = 0$ in $D_0 - m$, k being a constant, then u satisfies necessarily the equation also for the points of the set m. Therefore, u becomes analytic in the whole domain D_0 including even the set m.[1]

Proof. Our method for the proof of the proposition I follows that of the Lindeberg's theorem[2], suitably modified. Now, since we may suppose the set m, laid on the t-Plane, to be bounded, so we can cover it with a finite number of circles, $|t - a_\nu| < \rho_\nu$ $\nu = 1, 2, \cdots, n$, where, for any preassigned positive number ε, the ρ's satisfy a condition

$$(1) \quad \sum_{\nu=1}^{n} \frac{1}{|\log \rho_\nu|} < \varepsilon .$$

Remove all these circles from the domain D_0 and denote by D_ε the domain thus obtained. Let $C + C_\varepsilon$ be the boundary of the domain D_ε.

Now let us consider another function v which satisfies the equation $\Delta v + k^2 v = 0$ everywhere in D_0 including the set m, and which on C has the same boundary values as those of u. It is sufficient for the proof of our proposition to show that u coincides with v identically in $D_0 - m$. Now, let, by assumption, $|u| < k_1$ then the function $u - v$ has the following properties:

$$\Delta(u-v) + k^2(u-v) = 0 \quad \text{in} \quad D_\varepsilon$$

and

$$u - v = 0 \quad \text{on} \quad C ,$$

$$|u - v| < k_1 + k_1' = K \quad \text{on} \quad C_\varepsilon ,$$

since there exists a constant k_1' such that $|v| < k_1'$ on C_ε.

If we define a function w_ε by an equation

$$(2) \quad w_\varepsilon = K \sum_{\nu=1}^{n} \frac{Y_0(k|t - a_\nu|)}{\frac{1}{2}\log \rho_\nu} ,$$

where Y_0 denotes the Neumann's cylindrical function, then it is obvious that $\Delta w_\varepsilon + k^2 w_\varepsilon = 0$ in D_ε. And w_ε will behave as a majorant of $u - v$, that is, $w_\varepsilon > u - v$ in D_ε. In order to show this fact, we first investigate the boundary properties of the function w_ε. Let the distance between any point of C and any point of m be less than the number r_0/k where r_0 denotes the smallest positive zero-point of Y_0 and let $\rho_\nu < 1$, then $w_\varepsilon > 0$ on C, since $Y_0(k|t - a_\nu|)$ becomes negative in D_0.

What we can next say about boundary property of w_ε is that on C_ε $w_\varepsilon > K$. In fact, there holds a limit equation

$$\lim_{x \to 0} \frac{Y_0(kx)}{\frac{2}{\pi}\log x} = 1 ,$$

which implies

$$\frac{Y_0(kx)}{\frac{1}{2}\log x} > 1$$

for sufficiently small enough $|x|$. Hence we may suppose

$$\frac{Y_0(k\rho_\nu)}{\frac{1}{2}\log \rho_\nu} > 1$$

for points on the ν-th circle lying on C_ε, and further, remembering that D_0 is taken small enough,

$$\frac{Y_0(k|t - a_\mu|)}{\frac{1}{2}\log \rho_\mu} > 0 \qquad (\mu \neq \nu)$$

t on the ν-th circle.

Finally let us consider a function W_ε defined by $W_\varepsilon = w_\varepsilon - (u - v)$. It is readily seen that $\Delta W_\varepsilon + k^2 W_\varepsilon = 0$ in D_ε and $W_\varepsilon > 0$ on the boundary of D_ε, that is on $C + C_\varepsilon$. With help of a character of the first eigen-value as a domain function, we can conclude