ON THE SINGULARITIES OF THE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} + f(x,y)\frac{dy}{dx} + g(x,y) = P(x)$$

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§ 1

1. In this section we shall consider the differential equation

$$\text{(1)} \quad \frac{d^2y}{dx^2} + f(y)\frac{dy}{dx} + g(y) = P(x),$$

where f(y) and g(y) are polynomials of degree n and m respectively, i.e.,

$$f(y) = ay^n + by^{n-1} + \cdots + c, \quad a \neq 0$$

and

$$g(y) = \alpha y^m + \beta y^{m-1} + \cdots + \gamma, \quad \alpha \neq 0$$

and P(x) is a regular and single-valued function of x in certain neighborhood D of x^* on the x-plane. If we put $dy/dx = z$, we have a simultaneous equation

$$\begin{cases} \dfrac{dy}{dx} = z \\ \dfrac{dz}{dx} = P(x) - f(y)z - g(y). \end{cases}$$

Since the right hand side of it is regular in certain domain containing (x^*, y^*, z^*) in virtue of the hypotheses, there exists the one and only one regular solution through the point (x^*, y^*, z^*). If we continue the solution along a curve C, we may encounter a singular point or tend to the point at infinity. Hence the analytic continuation carries out a problem of singularities. In the sequel we shall exclusively consider a problem of isolated singularities which will appear as essential singularities, poles or branch points. And we always exclude the cases where $n = 0$ and $m = 0$ or 1.

We suppose that we can continue a solution $y = y(x)$ of (1) along any curve C up to a point x_0, but not beyond it. Further we suppose that, if we approach to x_0 along C, $y = y(x)$ tends to ∞. Then, the point $x = x_0$ is an isolated singularity and it may be a branch point. Then, we make a change of variable $x - x_0 = t^k$ if x_0 is finite and $x = t^{-k}$ if $x_0 = \infty$, where k is a positive integer not equal to zero and t is a local parameter which uniformize the solution in a neighborhood of x_0. Then, it follows from the equation (1) that

$$\text{(2)} \quad \frac{d^2y}{dt^2} + \left(kt^{k-1}f(y) - \frac{k-1}{t}\right)\frac{dy}{dt} + k^2t^{2(k-1)}g(y) = k^2t^{2(k-1)}P(x_0+t^k)$$

if $x - x_0 = t^k$, and

$$\text{(3)} \quad \frac{d^2y}{dt^2} + \left(\frac{k+1}{t} - \frac{kf(y)}{t^{k+1}}\right)\frac{dy}{dt} + \frac{k^2g(y)}{t^{2(k+1)}} = \frac{k^2P(t^{-k})}{t^{2(k+1)}}$$

if $x = t^{-k}$. According to the hypotheses, the solution of (2) is of the form

$$\text{(4)} \quad y = \sum_{\nu=-r}^{\infty} a_\nu t^\nu, \quad a_{-r} \neq 0, \quad r \geq 1.$$

Substituting (4) into (2), we obtain

$$\frac{r(r+1)a_{-r}}{t^{r+2}} + \cdots$$