SOME CLASSES OF POSITIVE SOLUTIONS OF $\Delta u = Pu$

ON RIEMANN SURFACES, II

By Mitsuru OZAWA

§ 4. Dimensions of ideal boundary (continuation).

If $F \in O_\Omega$ and $\Gamma \in (b)$, then the situation is somewhat troublesome to handle. Let $\mathcal{S}_v$ be the least harmonic majorant of $v \in \mathfrak{Q}_{F-\bar{F}_0}$. Then $\mathcal{S}_v$ has its sense for any $v \in \mathfrak{Q}_{F-\bar{F}_0}$. Let S_v be a limit harmonic function $\lim_{n\to\infty} v^n$ defined as follows: v^n is harmonic on $F_n - \bar{F}_0$ such that $v^n = v$ on $\Gamma_n + \Gamma_0$. Then we have easily that $\mathcal{S}_v \equiv S_v$. Let $\mathfrak{T}_u$ be the largest minorant of $u \in \mathbb{P}_{F-\bar{F}_0}$ belonging to $\mathfrak{P}_{F-\bar{F}_0}$. Then $\mathfrak{T}_u$ is equal to either constant zero or a solution of (A) such that $\mathfrak{T}_u \not\equiv 0$. If $\mathfrak{T}_u \not\equiv 0$, then $\mathfrak{T}_u \in \mathfrak{P}_{F-\bar{F}_0}$. Let u^n be a solution of (A) on $F_n - \bar{F}_0$ such that $u^n = u$ on $\Gamma_n + \Gamma_0$, then $T_u = \lim_{n\to\infty} u^n$ exists and either $T_u \equiv 0$ or $T_u \not\equiv 0$. Moreover T_u coincides with $\mathfrak{T}_u$. We have the following facts:

(i) $\mathfrak{T} \circ \mathcal{S} = I$ for any $v \in \mathfrak{Q}_{F-\bar{F}_0}$,

(ii) $\mathcal{S}$ operation preserves the linear independency.

Let $[U]$ be a positively linear subspace of $\mathbb{P}_{F-\bar{F}_0}$ spanned by all the minimals u_i, $i = 1, \cdots, n$ such that $\mathfrak{T}_{u_i} \not\equiv 0$, $u_i \in \mathbb{P}_{F-\bar{F}_0}$ and let $\mathfrak{T}[U]$ be T image of $[U]$. Let $[V]$ be a positively linear subspace of $\mathfrak{P}_{F-\bar{F}_0}$ for each element of which $\mathcal{S}$ operation has the sense. Evidently $\mathfrak{Q}_{F-\bar{F}_0} \subseteqq [V] \subset \mathfrak{P}_{F-\bar{F}_0}$. Let $\mathcal{S}[V]$ be $\mathcal{S}$ image of $[V]$. Next facts are also easy to verify:

$$\mathcal{S}[V] \subset [U] \text{ and } [V] \subset \mathfrak{T}[U].$$

Let u be a minimal in $\mathbb{P}_{F-\bar{F}_0}$ such that $\mathfrak{T}_u \not\equiv 0$. Then $\mathcal{S}\mathfrak{T}_u = u$ is valid. This shows that $\mathfrak{T}[U] \subset [V]$ and $[U] \subset \mathcal{S}[V]$. Hence we see that $[V] = \mathfrak{T}[U]$ and $\mathcal{S}[V] = [U]$.

Let u be a minimal belonging to $[U]$, then $\mathfrak{T}_u$ is also a minimal in $\mathfrak{P}_{F-\bar{F}_0}$. In fact, if we assume that $0 < w \leqq \mathfrak{T}_u$, then $\mathcal{S}_w$ exists and satisfies $0 < \mathcal{S}_w \leqq u$, therefore $\mathcal{S}_w = k u$ holds. This implies the desired fact $w = k\mathfrak{T}_u$.

If $F \in O_G^{(k)}$, then $u = \lim_{n\to\infty} g^{(k)}_{F-\bar{F}_0}(z, \zeta_n)$

for a suitable non-compact sequence $\{\zeta_n\}$ for any minimal u in $\mathbb{P}_{F-\bar{F}_0}$.

If $w = \lim_{m\to\infty} G_{F-\bar{F}_0}(z, \zeta_{n_m}) > 0$ on $F - \bar{F}_0$

for a suitable subsequence $\{\zeta_{n_m}\}$ of $\{\zeta_n\}$, then $0 < w \leqq \mathfrak{T}_u$ which shows that $w = k\mathfrak{T}_u$ and w is also a minimal in $\mathfrak{P}_{F-\bar{F}_0}$ and $\mathfrak{T}_u$ belongs to $\mathfrak{Q}_{F-\bar{F}_0}$.

Let $\mathcal{Q}_{F-\bar{F}_0}$ be a class of positive solutions v of (A) on $F - \bar{F}_0$ such that $0 < \iint_{F-\bar{F}_0} v(z) P(z)\, d\sigma < \infty$ and $v = 0$ on Γ_0. We shall next prove that S_v has the sense for any $v \in \mathcal{Q}_{F-\bar{F}_0}$. Evidently $v^n \geqq v$ on $F_n - \bar{F}_0$ and $v^n > v^m$ if $n > m$. Therefore $\frac{\partial v^n}{\partial \nu} > \frac{\partial v}{\partial \nu} \geqq 0$ on Γ_0 and $\frac{\partial v^n}{\partial \nu} \geqq \frac{\partial v}{\partial \nu}$ on Γ_n. On the other hand we see

$$\infty \gneqq M_1 > \int_{\Gamma_0} \frac{\partial}{\partial \nu} v\, ds + \iint_{F_n - \bar{F}_0} v P\, d\sigma$$

$$= - \int_{\Gamma_n} \frac{\partial}{\partial \nu} v\, ds > - \int_{\Gamma_n} \frac{\partial}{\partial \nu} v^n ds$$

$$= \int_{\Gamma_0} \frac{\partial}{\partial \nu} v^n ds ,$$

which leads to a fact that

$$M_1 \geqq \int_{\Gamma_0} \frac{\partial}{\partial \nu} S_v\, ds .$$

and hence we see that

$$S_v \not\equiv \infty$$