ON BOUNDARY VALUE PROBLEMS FOR A RECTANGLE

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1. Introduction.

In a series of preceding papers[1), 2), 3)] we have dealt with the transference between boundary value problems, Dirichlet's and Neumann's problems, for some domains of simple configuration. The leading idea of these papers is to reduce either of the problems to another by means of an elementary operation after suitably modifying the boundary functions. The method has been, indeed, once availed by Myrberg for the unit circle.[4)]

On the other hand, we have discussed a boundary value problem of mixed type for simply-connected domains and derived an explicit integral representation for the solution of the problem in case of a rectangle.[5)]

The purpose of the present paper is to show that the method of transference applies also to the mixed problem for a rectangle, by establishing a connection between this problem and an associated Dirichlet problem. In particular, an alternative way of deriving the formula will be implied for the solution of the mixed boundary value problem.

2. Theorems.

We now state our theorems explaining how the transference between the boundary value problems under consideration is to be performed. Theorem 1 concerns the transference from a mixed problem to Dirichlet problem, while theorem 2 concerns the transference of inverse order.

Theorem 1. Let, in the $z=x+iy$-plane, a Dirichlet problem for a basic rectangle

$$R:\ \lg q<x<0,\quad 0<y<\pi$$

with the boundary condition

$$u(it)=M(t),\quad u(\lg q+it)=N(t)\qquad (0<t<\pi),$$

$$u(s)=P(s),\quad u(s+i\pi)=Q(s)\qquad (\lg q<s<0)$$

be proposed, $M(t)$, $N(t)$, $P(s)$ and $Q(s)$ being supposed bounded and continuous in their respective intervals of definition. Solve by $w(z)$ (with bounded $\partial w(z)/\partial y$) an associated mixed boundary value problem with the boundary condition

$$w(it)=\int_0^t M(t)dt,\quad w(\lg q+it)=\int_0^t N(t)dt\qquad (0<t<\pi),$$

$$\frac{\partial w}{\partial \nu}(s)=P(s),\qquad \frac{\partial w}{\partial \nu}(s+i\pi)=-Q(s)\qquad (\lg q<s<0),$$

$\partial/\partial\nu$ designating the differentiation along inward normal. The solution $u(z)$ of the original Dirichlet problem is then given by

$$u(z)=\frac{\partial w(z)}{\partial y}.$$

Proof. Harmonicity of $u(z)$ follows immediately from that of $w(z)$. It is also immediate that the boundary condition for $u(z)$ along the horizontal sides $z=s$ and $z=s+i\pi$ $(\lg q<s<0)$ is fulfilled. That the boundary condition for $u(z)$ along the vertical sides $z=it$ and $z=\lg q+it$ $(0<t<\pi)$ is also fulfilled may be shown as follows. Let $w_0(z)$ be a function bounded and harmonic in the half-plane $x<0$ and satisfying the boundary