NOTE ON SIGULARITIES OF DIFFERENTIAL EQUATIONS

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In this note, Part I is devoted to study the general theory for critical points at the origin and in Part II special cases are treated. In both cases we shall stand on the view point of complex variables.

Part I

1. Let $f(z, w)$ be an analytic function of z and w in certain neighbourhood of the origin. Instead of analyticity we may suppose that $f(z, w)$ has the first continuous derivatives. Further we suppose that the Lipschitz condition holds good and $f(0, 0) = 0$. Then, we consider the equation

$$dz/dt = f(z, \bar{z}). \tag{1.1}$$

In virtue of the hypotheses the characteristics passing through $z = z_0$ $(\neq 0)$ is uniquely determined.

Critical points of the equation (1.1) are the points defined as $f(z, \bar{z}) = 0$. In the sequel we consider the properties of the characteristics of (1.1) in the neighbourhood of the origin $z = 0$.

By means of the hypotheses, in the neighborhood of the origin the equation (1.1) leads to the equation

$$\frac{dz}{dt} = f_n(z, \bar{z}) + f_{n+1}(z, \bar{z}),$$

where

$$f_n(z, \bar{z}) = a_0 z^n + a_1 z^{n-1}\bar{z} + \cdots + a_n \bar{z}^n$$

and $f_{n+1}(z, \bar{z})$ is the function with order at least $n+1$. Then, the properties of critical point at the origin is identical with the equation

$$dz/dt = f_n(z, \bar{z}). \tag{1.2}$$

2. <u>Indices</u>. Suppose that there exists no zero points of $f_n(z, 1)$ on the unit circle $|z| = 1$. Then, the origin is the only zero point of $f_n(z, \bar{z})$ and of course it is an isolated singularity. Hence, we can calculate its index.

Describe a circle of sufficiently small radius r with center at the origin. Generally let $f(z, \bar{z})$ be a function having the origin as an isolated zero point. Then, if we consider $f(z, \bar{z})$ as a vector defined in the neighborhood of the origin, the index of $z = 0$ for the function $f(z, \bar{z})$ is defined as

$$\text{Index of } z = 0 = I(0)$$

$$= \frac{1}{2\pi}\oint_{|z|=r} d \arg f(z, \bar{z}).$$

Hence, the index for the function $f_n(z, \bar{z})$ is

$$I(0) = \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(z, \bar{z}) \tag{1.3}$$

$$= n + \frac{1}{2\pi}\oint_{|z|=r} d \arg f_n(1, e^{-2i\theta}).$$

Let k be the number of zero points of $f_n(1, z)$ in the unit circle. Since $f_n(1, z)$ has no zero points on $|z| = 1$, we have by (1.3)

$$I(0) = n - 2k.$$

Indices are invariant under any regular transformation. Therefore, if we consider the equation only in the neighborhood of the origin, the equation (1.2) is topologically equivalent to the equation

$$\frac{dz}{dt} = z^{n-2k} \tag{1.4}$$

if $n \geqq 2k$, and