ON TRANSFERENCE OF BOUNDARY VALUE PROBLEMS

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Introduction.

In potential theory two kinds of boundary value problems, Dirichlet and Neumann problems, have been investigated among others in particular detail. Let D be a basic Jordan domain in the z-plane with a boundary contour C, along which a continuous boundary function $U(s)$ or $V(s)$ is assigned, s denoting the arc-length parameter. These problems may then be formulated as follows:

Determine a function $u(z)$ bounded and harmonic in D and satisfying the boundary condition

$$u = U(s) \quad \text{along} \quad C;$$

Determine a function $v(z)$ bounded and harmonic in D and satisfying the boundary condition

$$\partial v/\partial \nu = V(s) \quad \text{along} \quad C,$$

$\partial/\partial\nu$ denoting the differentiation along inward normal.

For Dirichlet problem the contour C may be quite arbitrary. But contrarily Neumann problem is, according to its own nature, usually considered with respect to a domain whose boundary contour is everywhere smooth. Further, while the solution of Dirichlet problem exists without any more restriction and is uniquely determined, the solution of Neumann problem exists if and only if its boundary function possesses the vanishing mean. When this condition for solvability is satisfied, the solution of Neumann problem is then unique except an arbitrary additive constant.

Let D be mapped one-to-one and conformally onto a Jordan domain $\mathfrak{D}$ with a boundary $\mathfrak{L}$ in the $\mathfrak{z}$-plane. The mapping yields then a continuous correspondence between the closed domain $D+C$ and $\mathfrak{D}+\mathfrak{L}$. Let the mapping function and its inverse be designated by $\mathfrak{z}=\mathfrak{z}(z)$ and $z=z(\mathfrak{z})$. Since Dirichlet problem is conformally invariant, the transformed function

$$\breve{u}(\mathfrak{z}) \equiv u(z(\mathfrak{z}))$$

solves the Dirichlet problem with the boundary condition

$$\breve{u} = U(s(\mathfrak{s})),$$

where $s=s(\mathfrak{s})$ designates a correspondence between the arc-length parameters on the boundaries induced by the mapping.

Suppose now that the mapping function possesses a continuous and non-vanishing derivative along the boundary. This is surely the case, for instance, provided both boundaries C and $\mathfrak{L}$ satisfy a Hölder condition of order greater than unity.[1] The Neumann problem is not purely invariant with respect to conformal mapping. However, the differential of its solution possesses the invariant character. In fact, the transformed function

$$\mathfrak{v}(\mathfrak{z}) \equiv v(z(\mathfrak{z}))$$

solves the Neumann problem with the boundary condition

$$\partial \mathfrak{v}/\partial \nu_{\mathfrak{z}} = V(s(\mathfrak{s}))\,|dz/d\mathfrak{z}|;$$

the condition for solvability is, of course, preserved:

$$0 = \int_C V(s)\,ds = \int_{\mathfrak{L}} V(s(\mathfrak{s}))\left|\frac{dz}{d\mathfrak{z}}\right| d\mathfrak{s}.$$

Let $G(z,\zeta)$ and $N(z,\zeta)$ be the Green function and Neumann function, respectively, of the domain D. The solutions of the original problems are then expressible by means of the well-known integral formulas