A THEOREM ON FOURIER SERIES

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Theorem.[1] Suppose that a function $f(x,t)$ is defined in $-\infty < x < \infty$, $0 \leqq t \leqq 1$, with period 2π and measurable with respect to x and there exists a function $S(x)$ with period 2π, integrable in $[0,2\pi]$ and, such that

$$|f(x,t)| \leqq S(x)$$

for every x and t. Now suppose further that $f(x,t)$ is continuous with respect to t at each x belonging to a set A ($\subset [0,2\pi]$) of positive measure; then $\sigma_n(x,t)$ tends to $f(x,t)$ uniformly in $0 \leqq t \leqq 1$ at almost all x belonging to A, where $\sigma_n(x,t)$ denotes the n-th Cesaro sum of the Fourier series of $f(x,t)$ of order 1 with respect to x.

Lemma.[2] If a function $f(x,t)$ defined in $a \leqq x \leqq b$, $0 \leqq t \leqq 1$ is measurable with respect to x for every t and continuous with respect to t for every x belonging to a set A^* of positive measure. Then for any positive number ε there exists a closed set $F^{(\varepsilon)}$ such that

$$m(A^* - F^{(\varepsilon)}) < \varepsilon$$

and $f(x,t)$ is continuous on

$$\{(x,t);\ x \in F^{(\varepsilon)},\ 0 \leqq t \leqq 1\}.$$

Proof. Let $t^{(1)}, t^{(2)}, \cdots, t^{(k)}, \cdots$ be all the rational numbers in $[0,1]$. Since $f(x,t)$ is uniformly continuous on $[0,1]$ as a function of t at every $x \in A^*$, we have

$$A^* = \sum_{m=1}^{\infty} A^*_{n,m},$$

where $A^*_{n,m}$ is the set of all x^* ($\in A^*$) such that for every $t^{(i)}$ and $t^{(j)}$ that satisfy $|t^{(i)} - t^{(j)}| < 1/m$,

$$|f(x^*, t^{(i)}) - f(x^*, t^{(j)})| < 1/n.$$

$A^*_{n,m}$ are clearly measurable.

Now we take the positive numbers ε_i ($i = 1, 2, \cdots$), such that We can take a positive integer m_n for every positive integer n, such that

$$m(A^* - A^*_{n,m_n}) < \varepsilon_n.$$

On the other hand, we can take for every integer $k > 0$ a measurable set A^*_k, such that $f(x,t^{(k)})$ is continuous on A^*_k and

$$m(A^* - A^*_k) < \varepsilon_k$$

by Lusin's theorem. Then we have

$$m(A^* - \prod_{k=1}^{\infty} A^*_k \cdot \prod_{n=1}^{\infty} A^*_{n,m_n})$$

$$= m\left(\sum_{k=1}^{\infty}(A^* - A^*_k) + \sum_{n=1}^{\infty}(A^* - A^*_{n,m_n})\right)$$

$$< \sum_{k=1}^{\infty} \varepsilon_k + \sum_{n=1}^{\infty} \varepsilon_n = \varepsilon/2. \qquad (1)$$

Since $\prod_{k=1}^{\infty} A^*_k \cdot \prod_{n=1}^{\infty} A^*_{n,m_n}$ is measurable, it contains a closed set $F^{(\varepsilon)}$, such that

$$m(\prod_{k=1}^{\infty} A^*_k \cdot \prod_{n=1}^{\infty} A^*_{n,m_n} - F^{(\varepsilon)}) < \varepsilon/2. \qquad (2)$$

Then it follows from (1) and (2) that

$$m(A^* - F^{(\varepsilon)}) < \varepsilon.$$

Now let us prove that $f(x,t)$ is continuous on the set $\{(x,t);\ x \in F^{(\varepsilon)}, 0 \leqq t \leqq 1\}$. Since the set of $t^{(j)}$ ($j = 1, 2, \cdots$) is dense in $[0,1]$ and $f(x,t)$ is continuous with respect to t at every $x \in A^*$, we have

$$|f(x,t_1) - f(x,t_2)| \leqq 1/n,$$

whenever $x \in A^*_{n,m}$, $0 \leqq t_1 \leqq 1$, $0 \leqq t_2 \leqq 1$ and $|t_1 - t_2| < 1/m$. Consequently $f(x,t)$ converges to $f(x,t_0)$ uniformly on $\prod_{n=1}^{\infty} A_{n,m_n}$ when t tends to an arbitrary number t_0, such that $0 \leqq t_0 \leqq 1$. Since furthermore there exists a sequence