A CARACTERIZATION OF THE MAXIMAL IDEAL

IN A FACTOR OF THE CASE (II_∞)

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It is well-known ([5; Theorem VIII] and [3; Theorem 2]) that any factor on a Hilbert space can be classified into the cases (I_n), (I_∞), (II_1), (II_∞) and (III). Von Neumann [6] proved that a factor is simple, that is, contains no (two sided) ideal, if and only if it belongs to one of the cases (I_n), (II_1) and (III). Moreover, Calkin [1] showed that the factor of case (I_∞) contains only one maximal ideal, and which consists of all completely continuous operators. Recently, Misonou [4; Theorem 3] proved that any rings of operators is weakly central, that is, there is a one-to-one correspondence between the maximal ideals in the ring and the maximal ideals in its center. As a corrollary of his theorem, we know that there exists only one maximal ideal in a factor of case (II_∞). In this note we shall characterize this maximal ideal. For simplicity, we consider only in a separable Hilbert space, but the proof remains true in the non-separable case by the use of the corresponding results due to Dixmier [2].

Theorem. Each factor M of case (II_∞) has a unique maximal ideal, and this is characterized by all those operators, $A \in M$ of which $[R(A)]$ are finite. Here, by $[R(A)]$ we shall denote the closure of the range of A.

Proof proceeds by the Calkin's principle and the use of the results on the rings of operators ([5] and [2]).

Let J be the set of all $A \in M$ such that $[R(A)]$ is finite, then it is well known that J is an ideal in M by [6, p. 25]. But for completeness, we shall trace his proof. Let A, B be in J, then $A+B \in J$ follows from the facts that $[R(A+B)] \subseteq [R(A)] \vee [R(B)]$ and the right hand side is also finite ([5; Lemma 7. 3. 5.]). Since $[R(A)] \geqq [R(AX)]$, $A \in J$ implies $AX \in J$ for any $X \in M$. Finally, by $[R(A)] \sim [R(A^*)]$ ([5; Lemma 6. 2. 1.]), we see that $A \in J$ implies $A^* \in J$. Hence $XA = (A^*X^*)^* \in J$, if $A \in J$. Now $[R(1)] = H$ is infinite, whence $I \notin J$, and there exists a non-zero projection $E \in M$ such that $[R(E)]$ is finite, so that $E \in J$. Consequently, we see that J is a non-trivial ideal in M. That J is unique, maximal and uniformly closed, will follow from the next paragraph.

Let K be any non-trivial ideal in M. Suppose that there exists a patially isometric operator $X \in M$ which gives the equivalency $H \sim [R(A)]$ ([5; Lemma 7. 2. 2.]). It is well-known that $H \ominus [f \in H ; Af = 0] = [R(A^*)]$ so that we obtain $[R(A^*)] \sim [R(A)] \sim H$, and this implies the existence of a patially isometric operator $Y \in M$ which gives the equivalency $H \sim [R(A^*)]$. Now, put $B = XAY^*$, then $B \in K$. Moreover, it is easily proved that B is a one-to-one transformation on the whole space H, by the definition of X and Y. Hence there exists the inverse $B^{-1} \in M$, so that $I = BB^{-1} \in K$, and this implies $K = M$. Thus we obtain $K \subseteq J$. This completes the proof.

Bibliography.

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