ON SOME BOUNDARY VALUE PROBLEM IN AN ANNULUS

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In this paper, we shall give a method to solve a boundary value problem in an annulus. Consider an annulus

$$N: \qquad 1 < |z| < R,$$

whose boundary consists of two circles

$$C_1: \quad z = e^{i\theta}, \qquad C_2: \quad z = Re^{i\theta}$$

Let $E_1 = \{E_{1n}\}$ and $E_2 = \{E_{2n}\}$ be sets of enumerable open circular arcs having no common parts on C_1 and C_2, respectively, and let their complementary sets be $\overline{E}_1 = C_1 - E_1$ and $\overline{E}_2 = C_2 - E_2$. We assume that $\overline{E}_1$ and $\overline{E}_2$ are of capacity zero. We divide $\{E_{in}\}$, $i=1,2$, into two classes, $\{A_{in}\}$ and $\{B_{in}\}$, in an arbitrary way.

Problem I: To construct a function $f(z)$ which is harmonic in the annulus N, and satisfies the following boundary conditions:

$$(1)\begin{cases} \lim\limits_{r\to 1} \dfrac{\partial f(re^{i\theta})}{\partial r} = A_1(\theta) & \text{for } e^{i\theta} \text{ on } \{A_{1n}\}, \\ \lim\limits_{r\to R} \dfrac{\partial f(re^{i\theta})}{\partial r} = \dfrac{1}{R}A_2(\theta) & \text{for } Re^{i\theta} \text{ on } \{A_{2n}\}, \\ \lim\limits_{r\to 1} f(re^{i\theta}) = B_1(\theta) & \text{for } e^{i\theta} \text{ on } \{B_{1n}\}, \\ \lim\limits_{r\to R} f(re^{i\theta}) = B_2(\theta) & \text{for } Re^{i\theta} \text{ on } \{B_{2n}\}, \end{cases}$$

where $A_i(\theta)$ and $B_i(\theta)$ are the given functions bounded and continuous on respective sets.

Now we replace our problem I in an annulus with an equivalent one in a half plane. For that purpose, we cut the annulus along the negative real axis, namely we restrict θ to vary within the interval $(-\pi, \pi)$. Let this annulus with the cut be N^*. By the conformal mapping

$$(2) \qquad t = \exp\{i\pi \log z/\log R\},$$

the annulus N^* is mapped onto the domain

$$D: \ \frac{1}{P} < \rho < P, \qquad v > 0,$$

where $t = \rho e^{i\varphi} = u + iv$ and $P = \exp(\pi^2/\log R)$. This domain D has two kinds of boundaries:

i) Segments of real axis,

$$v = 0, \quad \frac{1}{P} \leqq u \leqq P \ ;$$

$$v = 0, \quad -P \leqq u \leqq -\frac{1}{P},$$

which correspond to the circular boundaries of N^*.

ii) Two upper semi-circles

$$\rho = \frac{1}{P}, \qquad v > 0 \ ;$$

$$\rho = P, \qquad v > 0,$$

which correspond to the upper and lower banks of the cut respectively.

By the conformal mapping (2), our problem is transformed into the following one.

Problem II: To find a function $F(t)$ which is harmonic in D and satisfies the following boundary conditions:

$1°$. At the boundary points lying on the real axis, the boundary values of the function itself and of its normal derivative are equal to the values corresponding to the original ones, respectively,

$$\lim_{\varphi\to 0} \frac{\partial F(\rho e^{i\varphi})}{\partial\varphi} = \frac{\pi}{\log R} A_1\left(-\frac{1}{\pi}\log R \log\rho\right) \quad \text{for } u=\rho \text{ on } \{a_{1n}\},$$